New Number Fields
with known $p$-Class Tower

Conference: 22nd Czech and Slovak International Conference on Number Theory 2015
Place: Hotel Sorea Máj
Venue: Liptovský Ján, Slovakia
Date: Friday, September 04, 2015
Time: 10:00 – 10:20, a.m.
Author: Daniel C. Mayer (Austria)

A presentation within the frame of the international scientific research project

Towers of $p$-Class Fields over Algebraic Number Fields
ACKNOWLEDGEMENTS

I gratefully acknowledge that my research project is supported financially by the Austrian Science Fund (FWF): P 26008-N25.

I am indebted to M.F. Newman (ANU, Canberra, ACT) for pointing out that the capable groups $G$ in Theorems 3 and 7 have $p$-multiplicator rank $\mu(G) = 4$ and thus satisfy the inequality $d_2(G') \geq 2 + d_1(G')$ between their relation rank $d_2(G')$ and generator rank $d_1(G') = 2$. This disqualifies them as candidates for $p$-class tower groups of real quadratic fields, according to the Shafarevich Theorem in the Appendix.

Sincere thanks are given to Michael R. Bush (WLU, Lexington, VA) for making available brand-new (July 2015) numerical results on IPADs of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ and the distribution of discriminants $d > 0$ over these IPADs.

Finally, I thank Yasuhiro Kishi (AUE, Nagoya, JP) for our joint investigation of 5-class towers over certain cyclic quartic fields.
PREFACE

In this presentation, I use IPADs of 2nd order to show that real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with one of the 3-capitulation types c.18, (0122), and c.21, (2034), have a 3-class field tower of length 3.

These types are strange because they are unique with the following properties: Every finite metabelian 3-group $\mathcal{G}$ with one of these types $\kappa_1(\mathcal{G})$ has coclass $cc(\mathcal{G}) = 2$ and is infinitely capable with nuclear rank $\nu(\mathcal{G}) \geq 1$. So $\mathcal{G}$ cannot be leaf of a tree. The second 3-class group $Gal(F_3^2(K)|K)$ is such a group.

Section c (containing types 18 and 21) was found by Nebelung in 1989, 55 years after the Scholz/Taussky sections D,...,H. However, for 20 years nobody knew examples of fields with these types, which certainly cannot occur for complex quadratic fields, until I discovered suitable real quadratic fields, $d = 540{,}365$ with type c.21 on January 01, 2008, $d = 534{,}824$ with type c.18 on August 20, 2009.

After my return from the ICGA in Shanghai at the end of July 2015, I suddenly had the rewarding idea and the courage to study their 3-class tower.
INTRODUCTION.
SUCCINCT SURVEY

• Given a prime $p$, the Hilbert $p$-class field tower $F_p^\infty(K)$ is the maximal unramified pro-$p$ extension of an algebraic number field $K$.

• The key for determining its Galois group
  \[ G := G_p^\infty(K) = \text{Gal}(F_p^\infty(K)|K), \]
  which is briefly called the $p$-class tower group of $K$, is the structure of $p$-class groups $\text{Cl}_p(L)$ of unramified (abelian or non-abelian) extensions $L|K$.

• Our main intention is to present new criteria for the occurrence of assigned $p$-class tower groups $G$ and proofs of their actual realization by suitable base fields $K$.

• This presentation can be downloaded from http://www.algebra.at/22CSICNT2015.pdf
CHAPTER I.
THE GROUP THEORY
OF $p$-CLASS TOWER GROUPS
§ 0. The Artin Pattern of $G$

Let $p \geq 2$ be a prime number, $G$ a pro-$p$ group with commutator subgroup $G'$ and finite abelianization $G/G'$ of order $p^v$, $v \geq 1$.

**Definition 0.1.** (TTT, TKT and AP)

$Lyr_n(G) := \{G' \leq H \leq G \mid (G : H) = p^n\}$, $0 \leq n \leq v$, the $v + 1$ layers of intermediate normal subgroups between $G$ and $G'$.

$T_{G,H} : G/G' \to H/H'$ the Artin transfer [12] from $G$ to $H$,

$\tau_n(G) := (H/H')_{H \in Lyr_n(G)}$, $0 \leq n \leq v$,

the components of the multiple-layered Transfer Target Type (TTT) $\tau(G) := [\tau_0(G); \ldots; \tau_v(G)]$,

$\kappa_n(G) := (\ker(T_{G,H}))_{H \in Lyr_n(G)}$, $0 \leq n \leq v$,

the components of the multiple-layered Transfer Kernel Type (TKT) $\kappa(G) := [\kappa_0(G); \ldots; \kappa_v(G)]$.

The pair $AP(G) := (\tau(G'), \kappa(G'))$ is called the Artin pattern of $G$.

Definition 0.2. (IPAD)  
(N. Boston, M.R. Bush, F. Hajir, 2011 [5])

The *Index-p Abelianization Data* (IPAD) of $G$, 
\[ \tau^{(1)}(G') := [\tau_0(G'); \tau_1(G')] \]
arises by restriction of $\tau(G)$ to the $0^{th}$ and $1^{st}$ layer. It is a first order approximation of the TTT $\tau(G')$. Figure 1 shows a small non-trivial example of a multi-layered abelianization $G/G'$.

**Figure 1.** Layers of subgroups $G' \leq H_{i,j} \leq G$ for $G/G' = \langle x, y, G' \rangle \approx (p^2, p) \hat{=} 21$

Example: In the situation of Figure 1, the IPAD of $G$ is given by
\[ \tau^{(1)}(G) = [G/G'; (H_{1,1}/H'_{1,1}, \ldots, H_{1,p+1}/H'_{1,p+1})] \].
CHAPTER II.
NUMBER FIELDS WITH
3-CLASS TOWER OF LENGTH 3
§ 1. The Algorithm for Determining $G$

1st Step. (elementary abelian step)
We try to identify the metabelianization $\mathcal{G} := G/G''$ of the $p$-class tower group $G$ by means of the $p$-class groups $\text{Cl}_p(L)$ of unramified cyclic extensions $L|K$ of degree $p$.

2nd Step. (non-abelian step)
By computing the $p$-class groups $\text{Cl}_p(M)$ of unramified abelian extensions $M|L$ of increasing degrees $p, p^2, \ldots$, we are occasionally able to determine the $p$-class tower group $G$ [1, 2]. (The $M|K$ are also unramified but may be non-abelian.)

Figure 2, where $p = 3$, shows a simplified picture of the first few layers of unramified extensions of a quadratic field $K$ with $\text{Cl}_3(K) \simeq (3, 3) \hat{=} 1^2$ and $\text{Cl}_3(L_i) \simeq (3, 3, 3) \hat{=} 1^3$ for $i = 1, 2, 3$, $\text{Cl}_3(L_4) \simeq (9, 3) \hat{=} 21$. It is known that such a field is of type H.4 [9].

Figure 2. Layers of unramified extensions of a quadratic field $K$ of type H.4
§ 2. Notation

- $G$ a pro-$p$ group,
  $d_1(G) := \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ the *generator rank* of $G$,
  $d_2(G) := \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p)$ the *relation rank* of $G$.

- $\mathfrak{G}$ a finite $p$-group,
  $\nu(\mathfrak{G})$ the *nuclear rank* of $\mathfrak{G}$,
  $\mu(\mathfrak{G})$ the *$p$-multiplicator rank* of $\mathfrak{G}$.

If $\mathfrak{G}$ is metabelian (i.e. $\text{dl}(\mathfrak{G}) = 2$), then the set

$$\text{cov}(\mathfrak{G}) := \{\text{finite } G \neq \mathfrak{G} \mid G/G'' \simeq \mathfrak{G}\}$$

is called the *cover* of $\mathfrak{G}$.

- $K$ a number field,
  $\ell_p(K)$ the *length* of the $p$-class tower of $K$.

If $G = G_p^\infty(K)$, then $\ell_p(K) = \text{dl}(G)$.

In particular, if $\text{Cl}_3(K) \simeq (3, 3)^{\otimes 11}$ is the 3-class group of $K$, then $L_1, \ldots, L_4$ denote the unramified cyclic cubic extensions of $K$, and $\tau^{(1)}(K) = [\text{Cl}_3(K); \text{Cl}_3(L_1), \ldots, \text{Cl}_3(L_4)]$ is the 1st IPAD of $K$, according to the Artin Theorem in the Appendix.
§ 3. Capitulation Type c.18, Ground State

Assumptions:

$K$ a number field with $\text{Cl}_3(K) \simeq 11$,
$L_1, \ldots, L_4$ unramified cyclic cubic extensions of $K$,
$\kappa_1(K) = (0122)$ 3-capitulation type of $K$ in the $L_i$,
\[ \tau^{(1)}(K) = [11; 22, 111, 21, 21] \]
the 1$^{\text{st}}$ IPAD of $K$,
\[ \tau^{(2)}(K) = [11; (22; \tau_1(L_1)), (111; \tau_1(L_2)), (21; \tau_1(L_3)), (21; \tau_1(L_4))] \]
the 2$^{\text{nd}}$ IPAD of $K$, with $\tau_1(L_1) = ((21^2)^4)$.


$\mathfrak{G} = G/G'' \simeq \langle 3^6, 49 \rangle$ with $d_2(\mathfrak{G}) \geq 2 + d_1(\mathfrak{G})$,
and thus $\ell_3(K) \geq 3$.

Theorem 1. (D.C. Mayer, Aug. 2015 [15])

$\text{cov}(\langle 3^6, 49 \rangle) = \{\langle 3^7, 284 \rangle, \langle 3^7, 291 \rangle \}$ (2 groups)
and

1. $\tau_1(L_2) = ((21^2)^4, (1^2)^9), \tau_1(L_3) = \tau_1(L_4) = (21^2, (21)^3)$
\[ \iff G \simeq \langle 3^7, 284 \rangle. \]

2. $\tau_1(L_2) = (21^2, (1^3)^3, (1^2)^9), \tau_1(L_3) = (21^2, (21)^3),$
$\tau_1(L_4) = (21^2, (31)^3)$
\[ \iff G \simeq \langle 3^7, 291 \rangle. \]

In both cases, $d_2(G) = 1 + d_1(G)$, $dl(G') = 3$. 
§ 4. Real Quadratic Fields of Type c.18

Proposition. (D.C. Mayer, Feb. 2010 [9, 11])

In the range $0 < d < 10^7$ of fundamental discriminants $d$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ there exist precisely 28 cases with 3-capitulation type $\kappa_1(K) = (0122)$ and $\tau^{(1)}(K) = [11; 22, 111, (21)^2]$.

Theorem 2. (D.C. Mayer, Aug. 2015 [15])

1.

The 10 real quadratic fields (36%) with the following discriminants $d$,

$1 030 117, 3 259 597, 3 928 632, 4 593 673, 5 327 080, 5 909 813, 7 102 277, 7 738 629, 7 758 589, 9 583 736$,

have 3-class tower group $G \simeq \langle 3^7, 284 \rangle, \ell_3(K) = 3$.

2.

The 18 real quadratic fields (64%) with the following discriminants $d$,

$534 824, 2 661 365, 2 733 965, 3 194 013, 3 268 781, 4 006 033, 5 180 081, 5 250 941, 5 489 661, 6 115 852, 6 290 549, 7 712 184, 7 857 048, 7 943 761, 8 243 113, 8 747 997, 8 899 661, 9 907 837$,

have 3-class tower group $G \simeq \langle 3^7, 291 \rangle, \ell_3(K) = 3$. 
Figure 3 visualizes the groups in Theorems 1 and 3 and their population in Theorems 2 and 4.

**Figure 3.** Non-metabelian 3-tower groups $G$ on the pruned tree $\mathcal{T}_i((243, 6))$ [3]
§ 5. Capitulation Type c.18, Excited State

Assumptions:
\( \gamma_1(K) = (0122) \) 3-capitulation type of \( K \) in the \( L_i \),
\( \tau^{(1)}(K) = [11; 33, 111, 21, 21] \) the 1st IPAD of \( K \),
\( \tau^{(2)}(K) = [11; (33; \tau_1(L_1)), (111; \tau_1(L_2)), (21; \tau_1(L_3)), (21; \tau_1(L_4))] \) the 2nd IPAD of \( K \), with \( \tau_1(L_1) = ((321)^4) \).

The elementary abelian step of the algorithm yields:

**Proposition.** (D.C. Mayer, 2010 [9, 11] and 2015)
\( \mathfrak{G} = G/G'' \cong \langle 3^7, 285 \rangle - \#1; 1 \)
with \( d_2(\mathfrak{G}) \geq 2 + d_1(\mathfrak{G}) \), and thus \( \ell_3(K) \geq 3 \).

Next, the non-abelian step of the algorithm:
Theorem 3. (D.C. Mayer, Aug. 2015 [15])
\[
\text{cov}(\langle 3^7, 285 \rangle - \#1; 1) = \{
\langle 3^7, 285 \rangle - \#1; 1 - \#1; 7,
\langle 3^6, 49 \rangle - \#2; 1 - \#1; 1,
\langle 3^6, 49 \rangle - \#2; 1 - \#1; 1 - \#1; 1,
\langle 3^6, 49 \rangle - \#2; 1 - \#1; 1 - \#1; 8 \} \text{ (4 groups) and}
\]
1. \(\tau_1(L_2) = (321, (1^3)^3, (1^2)^9), \tau_1(L_i) = (321, (21)^3)\),
   \(i = 3, 4 \iff G \cong \langle 3^7, 285 \rangle - \#1; 1 - \#1; 7\),
2. \(\tau_1(L_2) = (321, (21^2)^3, (1^2)^9), \tau_1(L_i) = (321, (31)^3)\),
   \(i = 3, 4 \iff G \cong \langle 3^6, 49 \rangle - \#2; 1 - \#1; 1 - \#1; 8 \text{ or}
\]
Corollary. (D.C. Mayer, Aug. 2015)
If \(K\) has torsion free Dirichlet unit rank 1, then
1. \(\tau_1(L_2) = (321, (1^3)^3, (1^2)^9), \tau_1(L_i) = (321, (21)^3)\),
   \(i = 3, 4 \iff G \cong \langle 3^7, 285 \rangle - \#1; 1 - \#1; 7\).
2. \(\tau_1(L_2) = (321, (21^2)^3, (1^2)^9), \tau_1(L_i) = (321, (31)^3)\),
   \(i = 3, 4 \iff G \cong \langle 3^6, 49 \rangle - \#2; 1 - \#1; 1 - \#1; 8 \text{ or}
\]
§ 6. Real Quadratic Fields of Type c.18

**Proposition.** (M.R. Bush, Jul. 2015 [6, 7])

In the range $0 < d < 10^8$ of fundamental discriminants $d$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ there exist precisely 8 cases with 1st IPAD $\tau^{(1)}(K) = [11; 33, 111, (21)^2]$.

**Corollary.** (D.C. Mayer, 2010 [9, 11])

A quadratic field $K$ with $\tau^{(1)}(K) = [11; 33, 111, (21)^2]$ must be a real quadratic field with 3-capitulation type $\kappa_1(K) = (0122)$.

**Theorem 4.** (D.C. Mayer, Aug. 2015 [15])

1. The 4 real quadratic fields (50%) with the following discriminants $d$,
   
   13 714 789, 24 037 912, 54 683 977, 94 272 565,

   have 3-class tower group $G \simeq \langle 3^7, 285 \rangle - #1; 1 - #1; 7$.

2. The 4 real quadratic fields (50%) with the following discriminants $d$,
   
   14 252 156, 46 748 181, 67 209 369, 78 200 897,

   have 3-class tower group either
   
   $G \simeq \langle 3^6, 49 \rangle - #2; 1 - #1; 1 - #1; 1$ or
   
   $G \simeq \langle 3^6, 49 \rangle - #2; 1 - #1; 1 - #1; 8$.

In each case, the length of the 3-class tower of $K$ is given by $\ell_3(K) = 3$. 
§ 7. Capitulation Type c.21, Ground State

Assumptions:
\( \varkappa_1(K) = (2034) \) 3-capitulation type of \( K \) in the \( L_i \),
\( \tau^{(1)}(K) = [11; 21, 22, 21, 21] \) the 1st IPAD of \( K \),
\( \tau^{(2)}(K) = [11; (21; \tau_1(L_1)), (22; \tau_1(L_2)), (21; \tau_1(L_3)), (21; \tau_1(L_4))] \)
the 2nd IPAD of \( K \), with \( \tau_1(L_2) = ((21^2)^4) \).

\( \mathcal{G} = G/G'' \simeq \langle 3^6, 54 \rangle \) with \( d_2(\mathcal{G}) \geq 2 + d_1(\mathcal{G}) \),
and thus \( \ell_3(K) \geq 3 \).

Theorem 5. (D.C. Mayer, Aug. 2015 [15])
\( \text{cov}(\langle 3^6, 54 \rangle) = \{\langle 3^7, 307 \rangle, \langle 3^7, 308 \rangle\} \) (2 groups)
and
1. \( \tau_1(L_1) = \tau_1(L_3) = (21^2, (21)^3), \tau_1(L_4) = (21^2, (31)^3) \)
\( \iff G \simeq \langle 3^7, 307 \rangle \).
2. \( \tau_1(L_1) = \tau_1(L_4) = (21^2, (21)^3), \tau_1(L_3) = (21^2, (31)^3) \)
\( \iff G \simeq \langle 3^7, 308 \rangle \).

In both cases, \( d_2(G) = 1 + d_1(G), \text{dl}(G') = 3 \).

Remark. Since there is no natural ordering on the four maximal subgroups of the groups \( G \) in Theorem 5, the conditions of the two statements in this theorem are indistinguishable.

The two groups \( \langle 2187, 307 \rangle \) and \( \langle 2187, 308 \rangle \) have very similar properties and it is a challenge to find properties of a field \( K \) which allow its 3-class tower group \( G \) to be identified as an abstract group.
§ 8. Real Quadratic Fields of Type c.21

**Proposition.** (D.C. Mayer, Feb. 2010 [9, 11])

In the range $0 < d < 10^7$ of fundamental discriminants $d$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ there exist precisely 27 cases with 3-capitulation type $\kappa_1(K) = (2034)$. The 1st IPAD of 25 among them is $\tau^{(1)}(K) = [11; 21, 22, (21)^2]$, the remaining 2 have $\tau^{(1)}(K) = [11; 21, 33, (21)^2]$.

**Theorem 6.** (D.C. Mayer, Aug. 2015 [15])

The 25 real quadratic fields with the following discriminants $d$,

$540\,365, \quad 945\,813, \quad 1\,202\,680, \quad 1\,695\,260, \quad 1\,958\,629,$

$3\,018\,569, \quad 3\,236\,657, \quad 3\,687\,441, \quad 4\,441\,560, \quad 5\,512\,252,$

$5\,571\,377, \quad 5\,701\,693, \quad 6\,027\,557, \quad 6\,049\,356, \quad 6\,054\,060,$

$6\,274\,609, \quad 6\,366\,029, \quad 6\,501\,608, \quad 6\,773\,557, \quad 7\,573\,868,$

$8\,243\,464, \quad 8\,251\,521, \quad 9\,054\,177, \quad 9\,162\,577, \quad 9\,967\,837,$

have 3-class tower group either $G \cong \langle 3^7, 307 \rangle$ or $G \cong \langle 3^7, 308 \rangle$.

In both cases, the length of the 3-class tower of $K$ is given by $\ell_3(K) = 3$.

**Remark.** The remaining 2 discriminants are treated in Theorem 8.
Figure 4 visualizes the groups in Theorems 5 and 7 and their population in Theorems 6 and 8.

**Figure 4.** Non-metabelian 3-tower groups $G$ on the pruned tree $T_s((243, 8))$ [3]
§ 9. Capitulation Type c.21, Excited State

Assumptions:
κ₁(𝐾) = (2034) 3-capitulation type of 𝐾 in the 𝐿ᵢ,
τ⁽¹⁾(𝐾) = [11; 21, 33, 21, 21] the 1ˢᵗ IPAD of 𝐾,
τ⁽²⁾(𝐾) =
[11; (21; τ₁(𝐿₁)), (33; τ₁(𝐿₂)), (21; τ₁(𝐿₃)), (21; τ₁(𝐿₄))] the 2ⁿᵈ IPAD of 𝐾, with τ₁(𝐿₂) = ((321)^4).

The elementary abelian step of the algorithm yields:


\( \mathfrak{G} = G / G'' \simeq \langle 3^7, 303 \rangle - \#1; 1 \)
with \( d₂(\mathfrak{G}) \geq 2 + d₁(\mathfrak{G}) \), and thus \( ℓ₃(𝐾) \geq 3 \).

Next, the non-abelian step of the algorithm:
**Theorem 7.** (D.C. Mayer, Aug. 2015 [15])

\[
\text{cov}(\langle 3^7, 303 \rangle - \#1; 1) = \{
\langle 3^7, 303 \rangle - \#1; 1 - \#1; 7,
\langle 3^6, 54 \rangle - \#2; 3 - \#1; 1,
\langle 3^6, 54 \rangle - \#2; 3 - \#1; 1 - \#1; 1,
\langle 3^6, 54 \rangle - \#2; 3 - \#1; 1 - \#1; 8 \} \text{ (4 groups) and}
\]

1. \( \tau_1(L_i) = (321, (21)^3), i = 1, 3, 4 \)

\[ \iff \]

\( G \cong \langle 3^7, 303 \rangle - \#1; 1, |G| = 3^8, \text{ or} \]
\( G \cong \langle 3^7, 303 \rangle - \#1; 1 - \#1; 7, |G| = 3^9. \)

2. \( \tau_1(L_i) = (321, (31)^3), i = 1, 3, 4 \)

\[ \iff \]

\( G \cong \langle 3^6, 54 \rangle - \#2; 3 - \#1; 1, |G| = 3^9, \text{ or} \]
\( G \cong \langle 3^6, 54 \rangle - \#2; 3 - \#1; 1 - \#1; 7, |G| = 3^{10}, \)
\( G \cong \langle 3^6, 54 \rangle - \#2; 3 - \#1; 1 - \#1; 8, |G| = 3^{10}. \)

**Corollary.** (D.C. Mayer, Aug. 2015)

If \( K \) has torsion free Dirichlet unit rank 1, then

1. \( \tau_1(L_i) = (321, (21)^3), i = 1, 3, 4 \)

\[ \iff \]

\( G \cong \langle 3^7, 303 \rangle - \#1; 1 - \#1; 7. \)

2. \( \tau_1(L_i) = (321, (31)^3), i = 1, 3, 4 \)

\[ \iff \]

\( G \cong \langle 3^6, 54 \rangle - \#2; 3 - \#1; 1 - \#1; 7 \text{ or} \)
\( G \cong \langle 3^6, 54 \rangle - \#2; 3 - \#1; 1 - \#1; 8. \)
§ 10. Real Quadratic Fields of Type c.21↑

**Proposition.** (M.R. Bush, Jul. 2015 [6, 7])
In the range $0 < d < 10^8$ of fundamental discriminants $d$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ there exist precisely 12 cases with 1st IPAD $\tau^{(1)}(K) = [11; 21, 33, (21)^2]$.

**Corollary.** (D.C. Mayer, 2010 [9, 11])
A quadratic field $K$ with $\tau^{(1)}(K) = [11; 21, 33, (21)^2]$ must be a real quadratic field with 3-capitulation type $\kappa_1(K) = (2034)$.

**Theorem 8.** (D.C. Mayer, Aug. 2015 [15])
1. The 8 real quadratic fields (67%) with the following discriminants $d$,
   
   
   1 001 957, 9 923 685, 20 633 209, 58 650 717,
   63 404 792, 72 410 413, 84 736 636, 92 578 472,
   
   have 3-class tower group $G \simeq \langle 3^7, 303 \rangle - \#1; 1 - \#1; 7$.

2. The 4 real quadratic fields (33%) with the following discriminants $d$,
   
   25 283 701, 36 100 840, 42 531 528, 81 398 865,
   
   have 3-class tower group either
   
   $G \simeq \langle 3^6, 54 \rangle - \#2; 3 - \#1; 1 - \#1; 7$ or
   $G \simeq \langle 3^6, 54 \rangle - \#2; 3 - \#1; 1 - \#1; 8$.

In each case, the length of the 3-class tower of $K$ is given by $\ell_3(K) = 3$. 
CHAPTER III.
NUMBER FIELDS WITH
5-CLASS TOWER OF LENGTH 2
§ 11. Cyclic Quartic Fields

Let \( M = \mathbb{Q}(\sqrt{d}) \) be a cyclic quartic field where \( \zeta = \exp\left(\frac{1}{5}2\pi i\right) \) is a primitive fifth root of unity and \( d > 0 \) with \( \gcd(d, 5) = 1 \) is a real quadratic fundamental discriminant.

**Proposition.** (D.C. Mayer, Jun. 2012 [8, 13])

In the range \( 0 < d < 5000 \) of fundamental discriminants \( d \) of real quadratic fields \( K = \mathbb{Q}(\sqrt{d}) \) with \( \gcd(5, d) = 1 \) there exist precisely 37 cases such that the 5-dual field \( M = \mathbb{Q}(\sqrt{d}) \) of \( K \) has a 5-class group \( \text{Cl}_5(M) \) of type \((5, 5)\).

**Theorem 9.** (Y. Kishi, D.C. Mayer, Jul. 2015)

1. For the 7 real quadratic fields (19%) with the following discriminants \( d \),

\[
457, 501, 1996, 2573, 3253, 4189, 4957,
\]
the 5-dual field \( M \) of \( K \) has 5-capitulation type \( \kappa_1(M) = (124563) \), a 4-cycle with two fixed points, and 5-class tower group \( G \cong \langle 5^5, 11 \rangle \).

2. For the 5 real quadratic fields (14%) with the following discriminants \( d \),

\[
581, 753, 2296, 2829, 4553,
\]
the 5-dual field \( M \) of \( K \) has 5-capitulation type \( \kappa_1(M) = (123456) \), the identity permutation, and 5-class tower group \( G \cong \langle 5^5, 14 \rangle \) (O. Taussky, 1970).

In each case, the length of the 5-class tower of \( M \) is given by \( \ell_5(M) = 2 \), and \( d_2(G) = d_1(G) \).
Figure 5 visualizes the situation of a two-stage 5-class tower in Theorem 9.

**Theorem 10.** (Y. Kishi, D.C. Mayer, Jul. 2015) For the 2 real quadratic fields (5\%) with discriminants \( d \in \{4357, 4444\} \) the 5-dual field \( M \) of \( K \) has 5-capitulation type \( \kappa_1(M) = (000000) \), a constant with six total capitulations, and abelian 5-class tower group \( G \cong \langle 5^2, 2 \rangle \), \( \ell_5(M) = 1 \).
\( k_1 := \mathbb{Q}(\sqrt{d}), \ d > 0, \ \gcd(5, d) = 1, \ k_2 := \mathbb{Q}(\sqrt{5d}). \)

**Reflection Theorem for** \( p = 5. \) (Y. Kishi [14])

Relation between the 5-class ranks of \( k_1, k_2 \) and \( M: \)
\[
r_5(M) = r_5(k_1) + r_5(k_2) + 2 - \delta_1 - \delta_2,
\]
where \( 0 \leq \delta_i \leq 1 \) (for the precise definition see [14]).

Now let \( \text{Cl}_5(M) \cong (5, 5), \)
\( E_1, \ldots, E_6 \) unramified cyclic 5-extensions of \( M, \)
\[
F_5(w) := \langle \rho, \sigma \mid \rho^4 = \sigma^5 = 1, \rho^{-1}\sigma\rho = \sigma^w \rangle
\]
two Frobenius groups of order 20 with \( 2 \leq w \leq 3. \)

**Theorem 11.** (Y. Kishi, D.C. Mayer, Jul. 2015)
The properties of the absolute extensions \( E_i|\mathbb{Q} \) and the values of the invariants in the Reflection Theorem for the 37 cases in the Proposition are:

1. For the 2 cases with \( \ell_5(M) = 1 \) in Theorem 10, we have \( r_5(k_1) = 1, \ r_5(k_2) = 0, \ \delta_1 = 0, \ \delta_2 = 1, \) and
\[
\text{Gal}(E_i|\mathbb{Q}) \cong F_5(2) \text{ for } 1 \leq i \leq 6.
\]

2. For the other 35 cases, including the 12 cases of \( \ell_5(M) = 2 \) in Theorem 9, we have pairwise conjugate non-Galois extensions \( E_3 \cong E_4, \ E_5 \cong E_6, \)
\[
\text{Gal}(E_1|\mathbb{Q}) \cong F_5(2), \ \text{Gal}(E_2|\mathbb{Q}) \cong F_5(3), \ \text{Gal}(E_3|\mathbb{Q}) \cong F_5(3), \\
\text{Gal}(E_4|\mathbb{Q}) \cong F_5(3), \ \text{Gal}(E_5|\mathbb{Q}) \cong F_5(3), \ \text{Gal}(E_6|\mathbb{Q}) \cong F_5(3)
\]
\[
r_5(k_1) = 1, \ r_5(k_2) = 0, \ \delta_1 = 1, \ \delta_2 = 0 \\
\text{for } d \in \{1 996, \ 3 121, \ 3 129, \ 3 253\}, \\
r_5(k_1) = r_5(k_2) = \delta_1 = \delta_2 = 1 \text{ for } d = 4 504, \text{ and} \\
r_5(k_1) = r_5(k_2) = \delta_1 = \delta_2 = 0 \text{ otherwise.}
APPENDIX.
THE ARITHMETIC
OF $p$-CLASS TOWER GROUPS
§ 12. Capitulation of $p$-Classes

**Definition 12.1.**

$K$ a number field of $p$-class rank $r_p(K) = 2$, $L_1, \ldots, L_{p+1}$ its unramified cyclic extension fields of degree $p$, $j_i = j_{L_i|K} : \text{Cl}_p(K) \to \text{Cl}_p(L_i)$ the extension homomorphisms of $p$-classes.

The family $\kappa_1(K) = (\ker(j_i))_{1 \leq i \leq p+1}$ is called the $p$-capitulation type of $K$ [10, 16].

The family $\tau_1(K) = (\text{Cl}_p(L_i))_{1 \leq i \leq p+1}$ is called the $p$-class group type of $K$ [8, 9].

**Theorem 12.1.** (E. Artin, 1929 [12])

The $p$-capitulation type $\kappa_1(K)$, resp. $p$-class group type $\tau_1(K)$, of $K$ coincides with the first layer TKT $\kappa_1(G')$, resp. TTT $\tau_1(G)$, of the $n$th $p$-class group $G = G_p^n(K)$, for any $2 \leq n \leq \infty$.

**Table 1.** Class extension and transfer

$$
\begin{array}{ccc}
\text{Cl}_p(K) & \longrightarrow & \text{Cl}_p(L) \\
\text{Artin} & \uparrow & \uparrow \\
\text{isomorphism} & G/G' & \longrightarrow & H/H' & \text{isomorphism} \\
T_{G,H} & 
\end{array}
$$
§ 13. Relation Rank of $p$-Class Tower Groups

**Theorem 13.1.** (I.R. Shafarevich, 1964 [18])

$p \geq 2$ prime number, $K$ number field with signature $(r_1, r_2)$ and torsion free unit rank $r = r_1 + r_2 - 1$, $S$ finite set of places of $K$ not divisible by $p$, $\zeta$ primitive $p$th root of unity, $G := G_{S,p}^\infty(K) = \text{Gal}(F_{S,p}^\infty(K)|K)$ the Galois group of the maximal pro-$p$ extension $F_{S,p}^\infty(K)$ of $K$ which is unramified outside of $S$, $d_1 := \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ the generator rank of $G$, $d_2 := \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p)$ the relation rank of $G$. Then

$$d_1 \leq d_2 \leq \begin{cases} d_1 + r & \text{if } S \neq \emptyset \text{ or } \zeta \notin K, \\ d_1 + 1 & \text{if } S = \emptyset \text{ and } \zeta \in K. \end{cases}$$

**Corollary 13.1.** $K = \mathbb{Q}(\sqrt{d})$ quadratic field with discriminant $d$ and $S = \emptyset$, $G := G_p^\infty(K) = \text{Gal}(F_p^\infty(K)|K)$ the Galois group of the maximal unramified pro-$p$ extension $F_p^\infty(K)$ of $K$, i.e., the $p$-class tower group of $K$. Then

$$d_2 = d_1 \quad \text{if } (d < 0 \text{ and } p \geq 3 \text{ odd}),$$

$$d_1 \leq d_2 \leq d_1 + 1 \quad \text{if either } (d < 0 \text{ and } p = 2) \text{ or } d > 0.$$
References.

[1] D.C. Mayer,
Periodic sequences of $p$-class tower groups,
DOI 10.4236/jamp.2015.37090.
(International Conference on
Groups and Algebras 2015,
Shanghai, China, 21 July 2015.)

[2] D.C. Mayer,
Index-$p$ abelianization data of
$p$-class tower groups,
*Adv. Pure Math.* **5** (2015), no. 5, 286–313,
DOI 10.4236/apm.2015.55029,
Special Issue on Number Theory
and Cryptography.
(29th Journées Arithmétiques 2015,
University of Debrecen, Hungary, 09 July 2015.)

[3] D.C. Mayer,
Periodic bifurcations in
descendant trees of finite $p$-groups,
*Adv. Pure Math.* **5** (2015), no. 4, 162–195,
DOI 10.4236/apm.2015.54020,
Special Issue on Group Theory.
3-class field towers of exact length 3, 
*J. Number Theory* **147** (2015), 766–777, 
DOI 10.1016/j.jnt.2014.08.010.

*Heuristics for p-class towers of imaginary quadratic fields*, 

private communication, 11 July, 2015.

*Heuristics for p-class towers of real quadratic fields*, 
in preparation.
[8] D.C. Mayer,
The distribution of second $p$-class groups on coclass graphs,
DOI 10.5802/jtnb842.
(27th Journées Arithmétiques 2011,
Faculty of Mathematics and Informatics,
University of Vilnius, Lithuania, 01 Jul. 2011.)

[9] D.C. Mayer,
Principalization algorithm via class group structure,

[10] D.C. Mayer,
Transfers of metabelian $p$-groups,
*Monatsh. Math*. **166** (2012), no. 3–4, 467–495,
DOI 10.1007/s00605-010-0277-x.

The second $p$-class group of a number field,
DOI 10.1142/S179304211250025X.

[13] A. Azizi, M. Talbi, and D.C. Mayer, The group \( \text{Gal}(F_5^2(K)|K) \) for \( K = \mathbb{Q} \left( (\zeta_5 - \zeta_5^{-1}) \sqrt{d} \right) \) of type \((5,5)\), in preparation.


