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Index-p Abelianization Data (briefly: IPADs) of p-Class Tower Groups

Conference:	29th Journées Arithmétiques 2015
Place:	Debreceni Egyetem, Tanterem Tíz,
	Egyetemváros, Nagyerdö
Venue:	Debrecen, Magyarország
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Time:	15:00–15:20, p.m.
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A presentation within the frame of the international scientific research project

Towers of p-Class Fields over Algebraic Number Fields

Three main targets of this talk:

- **Class** c and **Coclass** r of the Metabelian p-Tower Group $\mathrm{G}_p^2(K)$ are determined by IPADs,
- **Polarization** and **Stabilization** of IPAD Components in Coclass Trees,
- First Criteria for p-Class Towers $\mathrm{F}_p^\infty(K)$ of **Length** $\geq$ **3** with **Odd** Prime p, using Iterated IPADs of 2nd Order.

This presentation can be downloaded from the URL

http://www.algebra.at/29JADebrecen.pdf

It is a succinct version of the article

[T2] D.C. Mayer,
Index-p abelianization data of
p-class tower groups,
Adv. Pure Math. **5** (2015), no. 5, 286–313,
DOI 10.4236/apm.2015.55029,
Special Issue on Number Theory
and Cryptography,

but the computational results
have been extended considerably.

CHAPTER I. THE GROUP THEORY OF p-CLASS TOWER GROUPS

§ 0. Abelian Invariants

$p \geq 2$ prime number,
$A \ldots$ finite abelian group, $|A|$ a power of p.

Thm. $A \simeq (\mathbb{Z}/p^{m_1}\mathbb{Z})^{r_1} \bigoplus \ldots \bigoplus (\mathbb{Z}/p^{m_s}\mathbb{Z})^{r_s}$,
for uniquely determined integers $r \geq 0$ (the p-rank),
$0 \leq s \leq r$, $r = r_1 + \ldots + r_s$, $m_1 < \ldots < m_s$.

Dfn. *Abelian Type Invariants* of A:

$$\left(\overbrace{p^{m_s}, \ldots, p^{m_s}}^{r_s \text{ times}}, \ldots, \overbrace{p^{m_1}, \ldots, p^{m_1}}^{r_1 \text{ times}}\right)$$

in *power* form,

$$\mathrm{ATI}(A) := (m_s^{r_s}, \ldots, m_1^{r_1})$$

in *logarithmic* form (exponents indicating iteration).

Example: $p = 3$, $A \simeq (9, 3, 3, 3, 3) \Longleftrightarrow \mathrm{ATI}(A) = (21^4)$, $A \simeq (81, 81, 27, 27) \Longleftrightarrow \mathrm{ATI}(A) = (4^2 3^2)$.

$G \ldots$ pro-p group with commutator subgroup G',
and finite abelianization $G^{\mathrm{ab}} := G/G'$.

Dfn. *Abelian Quotient Invariants* of G:

$$\mathrm{AQI}(G) := \mathrm{ATI}(G^{\mathrm{ab}}).$$

Example: For $p = 3$, we have for instance

$A \simeq (9, 3, 3, 3, 3) \iff \mathrm{ATI}(A) = (21^4)$,
$A \simeq (81, 81, 27, 27) \iff \mathrm{ATI}(A) = (4^2 3^2)$.

Dfn. p-group G, $|G| = p^e$. *Logarithmic order*:
$\mathrm{lo}(G) := e = \mathrm{cl}(G) + \mathrm{cc}(G)$, sum of class and coclass.
In particular, for a finite abelian p-group A with $\mathrm{ATI}(A) = (m_s^{r_s}, \ldots, m_1^{r_1})$, we have

$$\mathrm{lo}(A) = r_1 m_1 + \ldots + r_s m_s.$$

Special finite abelian p-groups:

- *Trivial* group: $A = 1$, $s, r = 0$, $\mathrm{ATI}(A) = (0)$.
- *Cyclic* groups: $A \simeq (p^{m_1})$,
  $s = r = r_1 = 1$, $\mathrm{ATI}(A) = (m_1)$.
- *Homocyclic* groups: $A \simeq (\overbrace{p^{m_1}, \ldots, p^{m_1}}^{r_1 \text{ times}})$,
  $s = 1$, $r = r_1 \geq 2$, $\mathrm{ATI}(A) = (m_1^{r_1})$.
- **Nearly homocyclic** groups:
  $A \simeq (\overbrace{p^{m_2}, \ldots, p^{m_2}}^{r_2 \text{ times}}, \overbrace{p^{m_1}, \ldots, p^{m_1}}^{r_1 \text{ times}})$,
  $s = 2$, $r = r_1 + r_2 \geq 2$, and $\mathbf{m_2 = m_1 + 1}$,
  $\mathrm{ATI}(A) = (m_2^{r_2}, m_1^{r_1})$, $e := \mathrm{lo}(A) =$
  $= r_1 m_1 + r_2 m_2 = r_1 m_1 + (r - r_1)(m_1 + 1)$,
  and thus by Euclidean division with remainder:

$$e = m_1 r + r_2, \; 1 \leq r_2 < r.$$

Dfn. For odd $p \geq 3$, and $\mathbf{r = p - 1}$, we write
$A =: \mathrm{A}(p, e)$, also admitting $0 \leq r_2 < p-1$ and $e \geq 0$.

Example: For $p = 3$,
including two degenerate cases $0 \le e \le 1$,
and considering homocyclic groups (with even e)
as special nearly homocyclic groups,
we have

$\mathrm{A}(3,0) \simeq 1 \hat{=} (0)$,
$\mathrm{A}(3,1) \simeq (3) \hat{=} (1)$,
$\mathrm{A}(3,2) \simeq (3,3) \hat{=} (1^2)$,
$\mathrm{A}(3,3) \simeq (9,3) \hat{=} (21)$,
$\mathrm{A}(3,4) \simeq (9,9) \hat{=} (2^2)$,
$\mathrm{A}(3,5) \simeq (27,9) \hat{=} (32)$,
$\mathrm{A}(3,6) \simeq (27,27) \hat{=} (3^2)$,
$\mathrm{A}(3,7) \simeq (81,27) \hat{=} (43)$, etc.

§ 1. The Artin Pattern

Let $p \geq 2$ be a prime number, G a pro-p group
with commutator subgroup G'
and finite abelianization G/G' of order p^v, $v \geq 1$.

Definition 1.1.

$\mathrm{Lyr}_n(G) := \{G' \leq H \trianglelefteq G \mid (G:H) = p^n\}$,
for $0 \leq n \leq v$, the $v+1$ *layers* of
intermediate normal subgroups between G and G'.

$T_{G,H} : G/G' \to H/H'$ the *Artin transfer* [1]
from G to H.

$\tau_n(G) := (\mathrm{ATI}(H/H'))_{H \in \mathrm{Lyr}_n(G)}$, for $0 \leq n \leq v$,
the components of the multi-layered
transfer target type (TTT) $\tau(G) := [\tau_0(G); \ldots ; \tau_v(G)]$.

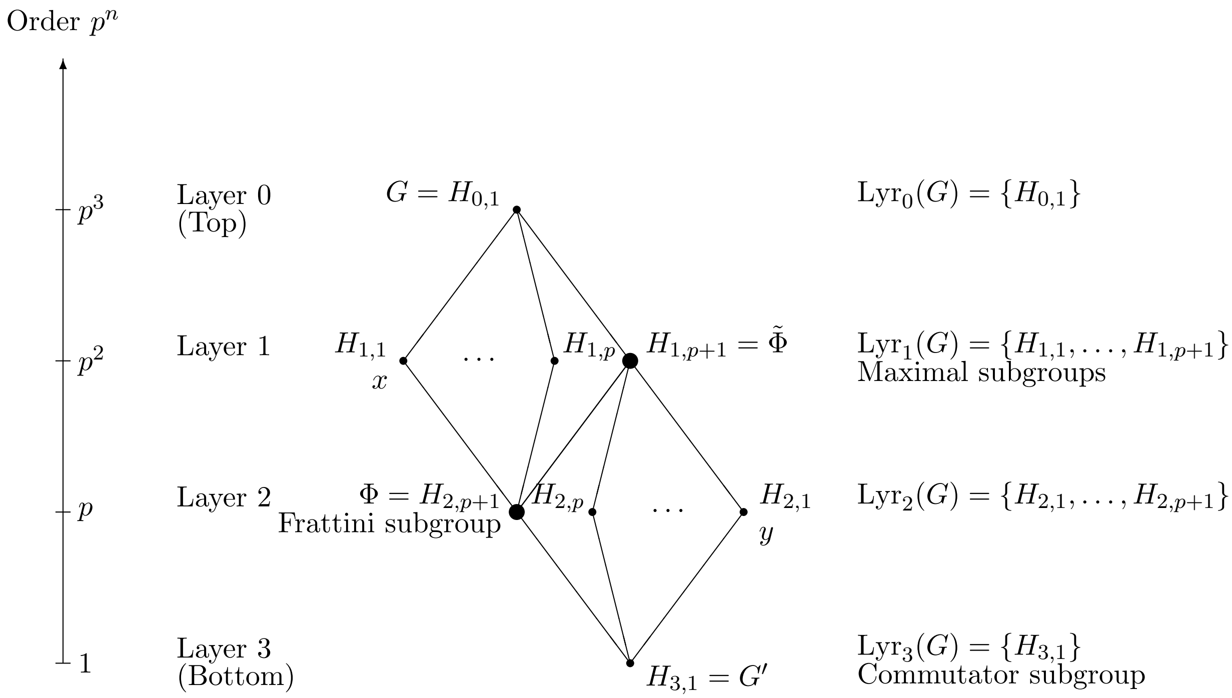
$\varkappa_n(G) := (\ker(T_{G,H}))_{H \in \mathrm{Lyr}_n(G)}$, for $0 \leq n \leq v$,
the components of the multi-layered
transfer kernel type (TKT) $\varkappa(G) := [\varkappa_0(G); \ldots ; \varkappa_v(G)]$.

The pair $\mathrm{AP}(G) := (\tau(G), \varkappa(G))$
is called the **Artin pattern** of G.

[1] E. Artin,
Idealklassen in Oberkörpern
und allgemeines Reziprozitätsgesetz,
Abh. Math.Sem. Univ.Hamburg **7** (1929), 46–51.

Figure 1 shows a small non-trivial example of a multi-layered abelianization G/G'.

FIGURE 1. Layers of subgroups $G' \le H_{i,j} \le G$ for $G/G' = \langle x, y, G' \rangle \simeq (p^2, p)$



Definition 1.2.
(N. Boston, M.R. Bush, F. Hajir, 2011)

The *Index-p Abelianization Data* (IPAD) of G,

$$\tau^{(1)}(G) := [\tau_0(G); \tau_1(G)],$$

arises by restriction to the zeroth and first layer.
It is a first order approximation of the TTT $\tau(G)$.

Example: In the situation of Figure 1, the IPAD of G is given by

$$\tau^{(1)}(G) = [\mathrm{ATI}(G/G'); (\mathrm{ATI}(H_{1,1}/H'_{1,1}), \dots, \mathrm{ATI}(H_{1,p+1}/H'_{1,p+1}))].$$

[BBH] N. Boston, M. R. Bush and F. Hajir,
Heuristics for p-class towers of imaginary quadratic fields,
to appear in Math. Annalen, 2015.
(arXiv: 1111.4679v2 [math.NT] 10 Dec 2014.)

§ 2. All IPADs for G/G' of Type $(3,3)$

Let $\mathfrak{G} := G/G''$ be the metabelianization of the pro-3 group G with abelianization G/G' of type $(3,3)$.

Definition 2.1. $\gamma_1(\mathfrak{G}) := \mathfrak{G}$, $\gamma_j(\mathfrak{G}) := [\gamma_{j-1}(\mathfrak{G}), \mathfrak{G}]$ for $j \geq 2$, the lower central series of $\mathfrak{G}$,
$c := \mathrm{cl}(\mathfrak{G})$ minimal such that $\gamma_c(\mathfrak{G}) > \gamma_{c+1}(\mathfrak{G}) = 1$,
$\gamma_2(\mathfrak{G}) \leq \chi_j(\mathfrak{G}) \leq \mathfrak{G}$ for $j \geq 2$, the jth *two-step centralizer* of $\mathfrak{G}$, maximal such that

$$[\chi_j(\mathfrak{G}), \gamma_j(\mathfrak{G})] \leq \gamma_{j+2}(\mathfrak{G}),$$

$s := \min\{j \geq 2 \mid \chi_j(\mathfrak{G}) > \gamma_2(\mathfrak{G})\}$ with $2 \leq s \leq c$, $e := \min\{j \geq 2 \mid \gamma_{j+1}(\mathfrak{G})/\gamma_{j+2}(\mathfrak{G}) \text{ cyclic}\}$ with $2 \leq e \leq c$. Then the *defect of commutativity* of $\mathfrak{G}$ is given by $0 \leq k = k(\mathfrak{G}) \leq 1$ such that $[\chi_s(\mathfrak{G}), \gamma_e(\mathfrak{G})] = \gamma_{c+1-k}(\mathfrak{G})$.

Theorem 2.1. The *defect of commutativity* $k = k(\mathfrak{G})$ of $\mathfrak{G}$ is given (phenomenologically) by

$$k = \begin{cases} 0 & \text{if } \mathfrak{G} \text{ has an abelian maximal subgroup,} \\ 1 & \text{if all maximal subgroups of } \mathfrak{G} \text{ are non-abelian,} \end{cases}$$

when $\mathfrak{G}$ is of coclass $r := \mathrm{cc}(\mathfrak{G}) = 1$, and by

$$k = \begin{cases} 0 & \text{if the centre } \zeta_1(\mathfrak{G}) \text{ is bicyclic of type } (3,3), \\ 1 & \text{if the centre } \zeta_1(\mathfrak{G}) \text{ is cyclic of order } 3, \end{cases}$$

when $\mathfrak{G}$ is of coclass $r = \mathrm{cc}(\mathfrak{G}) \geq 2$.

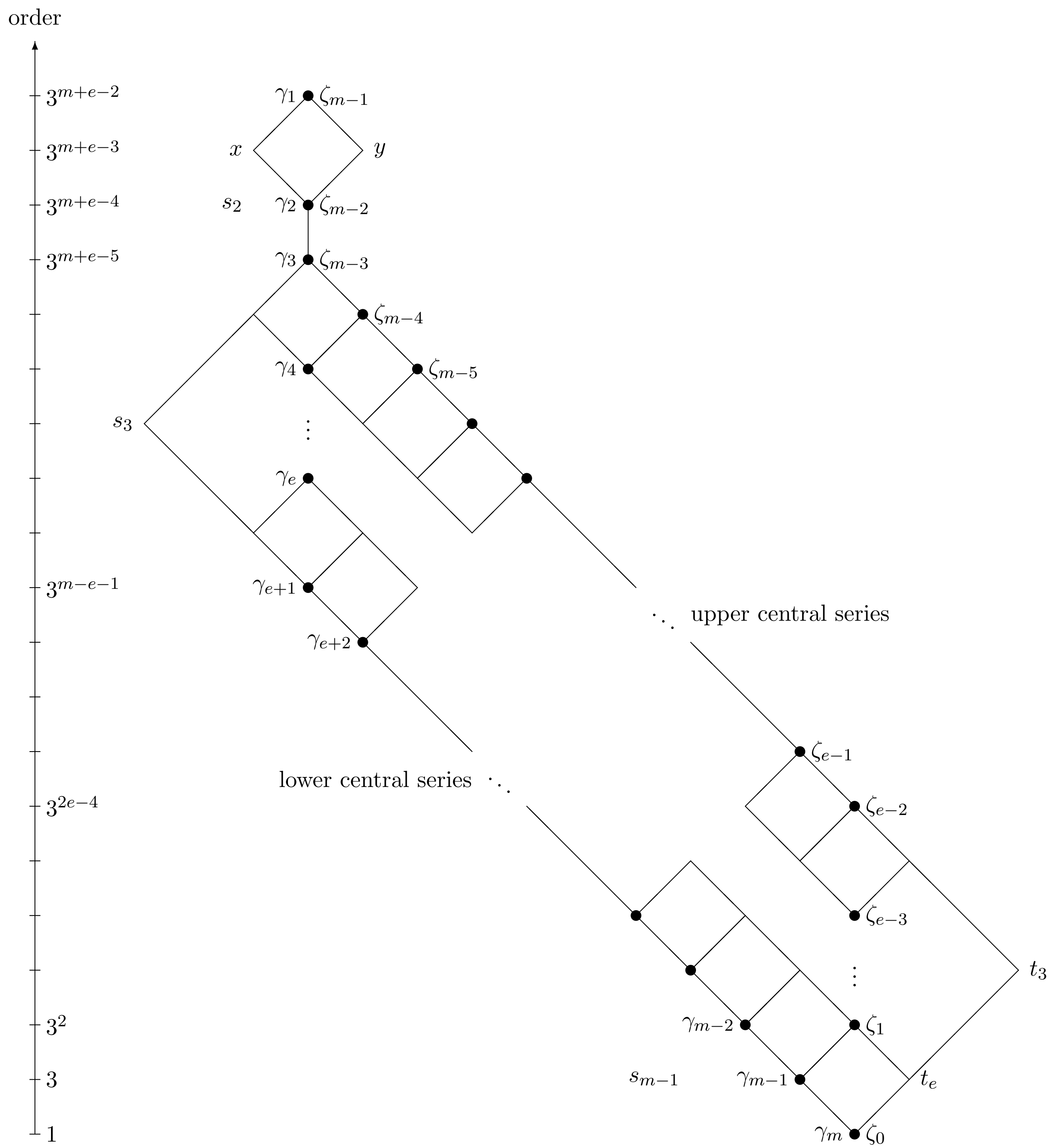
The distinction between coclass $r = 1$ and coclass $r \geq 2$ in Theorem 2.1 is necessary, since, under the given conditions, a group of maximal class always has a cyclic centre, and a group of non-maximal class never possesses an abelian maximal subgroup. The first part is due to Blackburn, the second part to Nebelung.

[Bl1] N. Blackburn,
On a special class of p-groups,
Acta Math. **100** (1958), 45–92.

[Ne] B. Nebelung,
Klassifikation metabelscher 3-*Gruppen mit Faktorkommutatorgruppe vom Typ* $(3,3)$ *und Anwendung auf das Kapitulationsproblem*,
Inauguraldissertation, Universität zu Köln, 1989.

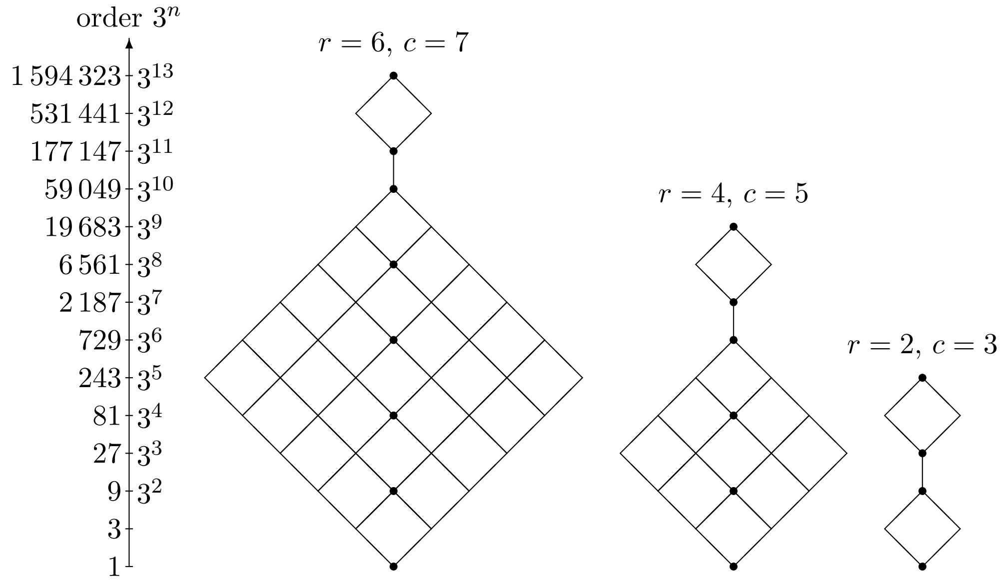
Figures 2–6 show the rectangular normal lattices for metabelianizations $\mathfrak{G}$ having defect $k(\mathfrak{G}) = 0$.

Figure 2. Normal lattice and central series of $\mathfrak{G}$ with $m = c + 1$, $e = r + 1$



Trailing diamonds for $3 \le j \le c$, $3 \le \ell \le r+1$:
$P_{j,\ell} = \Sigma_j \times T_\ell = \langle s_j, \ldots, s_c \rangle \times \langle t_\ell, \ldots, t_{r+1} \rangle$
$= \langle s_j, t_\ell, P_{j+1,\ell+1} \rangle$ in Figure 2.

FIGURE 3. Square normal lattice of groups $\mathfrak{G}$ with minimal class $c = r+1$



In Figure 3, we can see the square normal lattice of *interface groups* [MT4].
The last case, $r = 2$, $c = 3$, must be excluded from the following Main Theorem.

FIGURE 4. Rectangular normal lattice of groups $\mathfrak{G}$ with class $c \ge r+2$

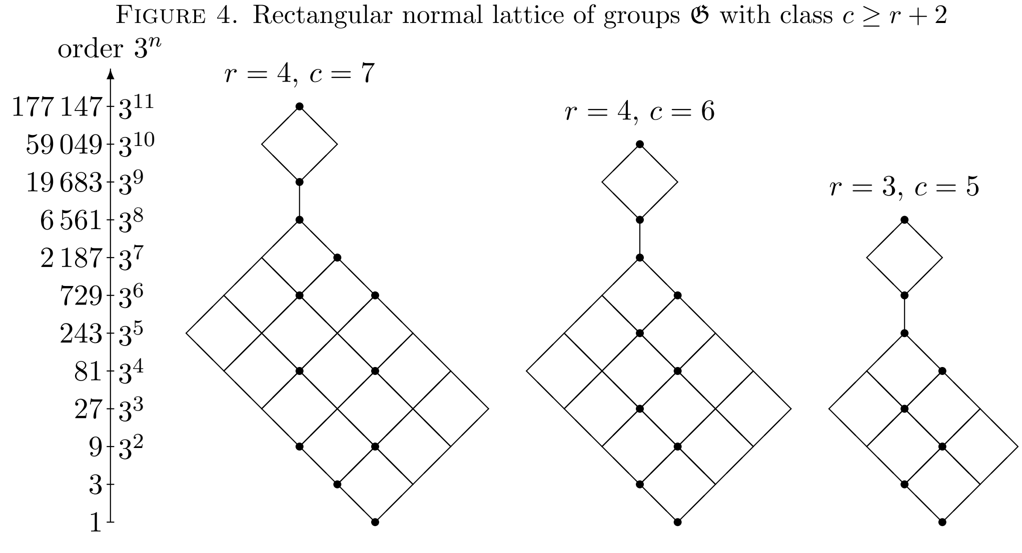


FIGURE 5. Rectangular normal lattice of groups $\mathfrak{G}$ with $r=2$, $c\geq 4$

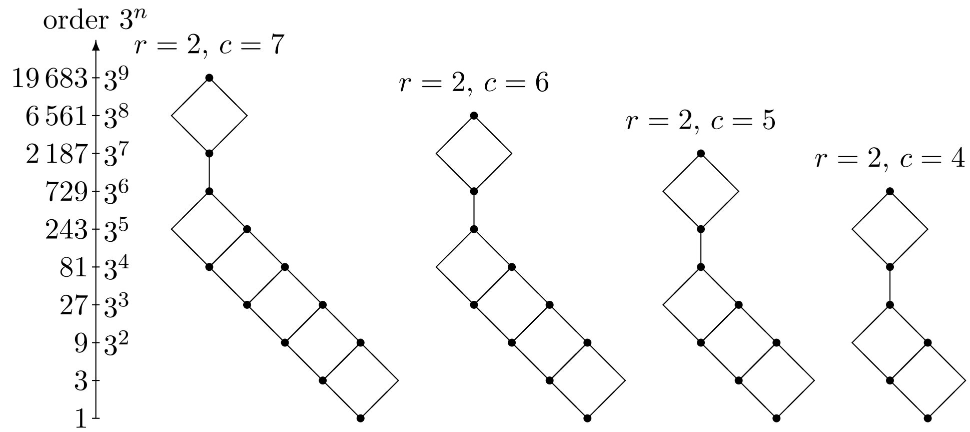
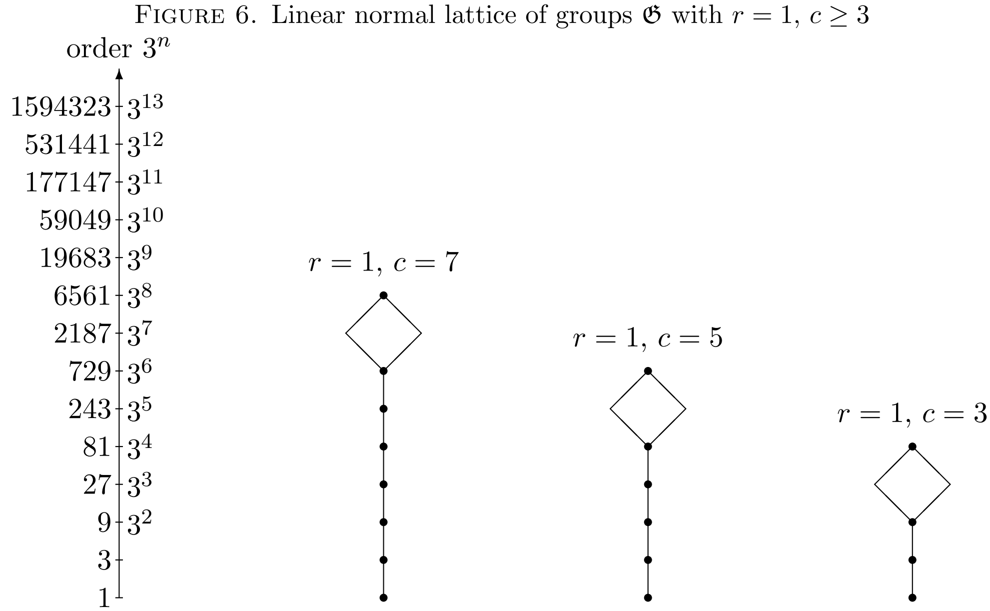


Figure 6 shows the linear normal lattice of groups of *maximal class* [MT2].
The last case, $r=1$, $c=3$, must be excluded from the following Main Theorem.

FIGURE 6. Linear normal lattice of groups $\mathfrak{G}$ with $r=1$, $c\geq 3$



Main Theorem 2.2. (Mayer, 2014)
Let G be a pro-3 group with abelianization G/G' of type $(3,3)$ and metabelianization $\mathfrak{G} = G/G''$.

Then the IPAD $\tau^{(1)}(G) = [\tau_0(G); \tau_1(G)]$ and the TTT $\tau(G) = [\tau_0(G); \tau_1(G); \tau_2(G)]$ are given in terms of nearly homocyclic abelian 3-groups by

$$\tau_0(G) = 1^2;$$

$$\tau_1(G) = \left(\overbrace{\mathrm{A}(3, c-k)}^{\text{polarization}}, \overbrace{\mathrm{A}(3, r+1)}^{\text{bipolarization}}, T_3, T_4 \right);$$

$$\tau_2(G) = \overbrace{\mathrm{A}(3, c-1)}^{\text{class factor}} \times \overbrace{\mathrm{A}(3, r-1)}^{\text{coclass factor}},$$

if either $r := \mathrm{cc}(\mathfrak{G}) = 1$ and $c := \mathrm{cl}(\mathfrak{G}) \geq 4$
or $r = 2$ and $c \geq 5$ or $c = 4$, $k = 0$,
or $r \geq 3$ and $c \geq r + 1$ except for $c = r + 2$, $k = 1$.

Here, the stabilizations (stable components) are $(T_3, T_4) =$

$$\begin{cases} (\mathrm{A}(3, r+1)^2) & \text{if } r = 2,\ \mathfrak{G} \in \mathcal{T}^2\langle 729, 54\rangle \text{ or } r = 1, \\ (1^3, \mathrm{A}(3, r+1)) & \text{if } r = 2,\ \mathfrak{G} \in \mathcal{T}^2\langle 729, 49\rangle, \\ ((1^3)^2) & \text{if } r = 2,\ \mathfrak{G} \in \mathcal{T}^2\langle 729, 40\rangle \text{ or } r \geq 3. \end{cases}$$

[MT3] D.C. Mayer,
Principalization algorithm
via class group structure,
J. Théor. Nombres Bordeaux **26** (2014),
no. 2, 415–464.

CHAPTER II. THE ARITHMETIC OF p-CLASS TOWER GROUPS

§ 3. Capitulation of p-Classes

Definition 3.1.
K a number field of p-class rank $r_p(K) = 2$,
$L_1, \ldots, L_{p+1}$
its unramified cyclic extension fields of degree p,
$j_i = j_{L_i|K} : \mathrm{Cl}_p(K) \to \mathrm{Cl}_p(L_i)$
the extension homomorphisms of p-classes.

The family $\varkappa_1(K) = (\ker(j_i))_{1 \le i \le p+1}$
is called the *p-capitulation type* of K [8], [MT2].

The family $\tau_1(K) = (\mathrm{ATI}(\mathrm{Cl}_p(L_i)))_{1 \le i \le p+1}$
is called the *p-class group type* of K [MT4].

Theorem 3.1. (Artin, 1929 [1])
The p-capitulation type $\varkappa_1(K)$, resp. p-class group type $\tau_1(K)$, of K coincides with the first layer TKT $\varkappa_1(G)$, resp. TTT $\tau_1(G)$, of the nth p-class group $G = \mathrm{G}_p^n(K)$, for any $2 \le n \le \infty$.

$$\begin{array}{ccccc} & & j_{L|K} & & \\ & \mathrm{Cl}_p(K) & \longrightarrow & \mathrm{Cl}_p(L) & \\ \text{Artin} & \updownarrow & & \updownarrow & \text{Artin} \\ \text{isomorphism} & G/G' & \longrightarrow & H/H' & \text{isomorphism} \\ & & T_{G,H} & & \end{array}$$

Theorem 3.2. (Combination of theorems by Herbrand, 1932, and Iwasawa, 1956)
$p \geq 3$ odd prime,
K number field of p-class rank $r_p(K) \geq 1$,
$L|K$ unramified cyclic extension of degree p. Then

$$\# \ker(j_{L|K}) = (U_K : \mathrm{Norm}_{L|K} U_L) \cdot [L : K].$$

Corollary 3.2. (Order of a single kernel)
In particular, if $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field, with fundamental unit $\eta > 1$ for $d > 0$, then

$$\# \ker(j_{L|K}) = \begin{cases} p & \text{if } d < 0 \text{ or } d > 0,\ \eta \in \mathrm{Norm}_{L|K} U_L, \\ p^2 & \text{if } d > 0,\ \eta \notin \mathrm{Norm}_{L|K} U_L. \end{cases}$$

Totally real dihedral fields L of these two kinds are called of type δ, resp. α, in the sense of Moser, 1975.

[Hb] J. Herbrand, Sur les théorèmes du genre principal et des idéaux principaux, *Abh. Math.Sem. Univ.Hamburg* **9** (1932), 84–92.

[Iw] K. Iwasawa, A note on the group of units of an algebraic number field, *J. Math. Pures Appl.* **35** (1956), 189–192.

[Mo] N. Moser, Unités et nombre de classes d'une extension Galoisienne diédrale de $\mathbb{Q}$, *Abh. Math.Sem.Univ.Hamburg* **48** (1979), 54–75.

In 1934, Scholz and Taussky [8] have established a terminology for 3-**capitulation types** $\varkappa_1(K)$ of **complex** quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d < 0$ and 3-class group $\mathrm{Cl}_3(K)$ of type $(3, 3)$, arranging similar types in **sections** denoted by upper case letters.

Concrete realizations of these 3-capitulation types for $d < 0$ have been found successively in 1934 [8], in 1982 [5], in 1984 [2], and the last two missing ones in 2003 [MT1].

However, for $d > 0$ it was believed from 1982 [5] until 2006 [MT1] that only three other 3-capitulation types, introduced by Nebelung in 1989, containing three or four total capitulations in fields of Moser's type α, can occur.

With considerable surprise, we discovered in the time between 2006 and 2010 that **real** quadratic fields also admit the 3-capitulation types of Scholz and Taussky, though with very low density of occurrences [MT1].

Table 1 gives a survey of those 3-capitulation types for which complete or at least partial results concerning the associated 3-**class field tower** are known currently [J1,T1,T2,T3].

Table 1. 3-capitulation types for $K = \mathbb{Q}(\sqrt{d})$, $\mathrm{Cl}_3(K) \simeq (3,3)$

Type			Minimal discriminant	
Section	No.[7]	$\varkappa_1(K)$	$d < 0$	$d > 0$
3-class tower known since 2009 (?): [Bl2]				
a [Ne]	3	(2000)	impossible	32 009 [5]
a [Ne]	2	(1000)	impossible	72 329 [5]
a [Ne]	1	(0000)	impossible	62 501 [5]
3-class tower well-known since 1934: [8,5,2,3]				
D [8]	10	(3144)	−4 027 [8]	422 573 [MT1]
D [8]	5	(1133)	−12 131 [5]	631 769 [MT1]
3-class tower controversial from 1934 until 2012: [2,3]				
E [8]	9	(2334)	−9 748 [8]	342 664 [MT1]
E [8]	8	(2234)	−34 867 [MT1]	6 098 360 [MT1]
E [8]	14	(3122)	−16 627 [5]	3 918 837 [MT1]
E [8]	6	(1122)	−15 544 [5]	5 264 069 [MT1]
3-class tower completely unknown up to 2015:				
G [8]	19	(2143)	−12 067 [5]	214 712 [MT1]
H [8]	4	(4111)	−3 896 [5]	957 013 [MT1]

[8] Scholz and Taussky, 1934
[5] Heider and Schmithals, 1982
[Ne] Nebelung, 1989
[7] Mayer, 1991
[MT1] Mayer, 2012

[Bl2] N. Blackburn,
On prime-power groups
in which the derived group has two generators,
Proc. Camb. Phil. Soc. **53** (1957), 19–27.

Statistical results concerning the most frequent 3-capitulation types are given in Table 2.

With 46.3%, towers of length 2, associated with the capitulation types D.10 and D.5, are clearly **dominating** in the range $-10^6 < d < 0$ of negative quadratic fundamental discriminants. This tendency remains rather stable in the extended ranges $-10^7 < d < 0$ and $-10^8 < d < 0$.

The **second** largest proportion of 20.3% is formed by towers of length 3, corresponding to capitulation types in section E, that is E.8, E.9 and E.6, E.14. Their existence was proved very recently in [J1,T3].

The most mysterious part of the population up to 2015 consisted of the capitulation type H.4, for which the length of the 3-class tower was completely open and could even have been infinite. With 14.7% it is located on the **third** place of all frequencies. Theorem 7.2 will remove the uncertainty about H.4.

Table 2. Classification of $K=\mathbb{Q}(\sqrt{d})$, $\mathrm{Cl}_3(K)\simeq(3,3)$, via IPADs

Type	$-10^6<d<0$	$-10^7<d<0$	$-10^8<d<0$
3-class tower of length 2: [8,5,2,3]			
D.10	667	7 622	83 353
D.5	269	3 625	41 398
subtotal	936	11 247	124 751
	46.3 %	46.0 %	45.1 %
3-class tower open till 2015:			
H.4	297	3 619	40 968
	14.7 %	14.8 %	14.8 %
G.19	94	1 019	10 426
total	1 327	15 885	176 145
	65.7 %	64.9 %	63.7 %
3-class tower of length 3: [J1,T3]			
E.8 or 9	210		
E.6 or 14	201		
subtotal	411		
	20.3 %		
among	2 020	24 476	276 375

3-class tower group $G:=\mathrm{G}_p^\infty(K)$ of $K=\mathbb{Q}(\sqrt{d})$ for $d<0$:
$\langle 243,5\rangle$ for type D.10,
$\langle 243,7\rangle$ for type D.5,
$\langle 729,54\rangle-\#2;2|4|6$ for 197 among 210 cases of types E.8 or 9,
$\langle 729,49\rangle-\#2;4|5|6$ for 186 among 201 cases of types E.6 or 14.

§ 4. Relation Rank of p-Class Tower Groups

Theorem 4.1. (Shafarevich, 1964 [9])
$p \geq 2$ prime number, K number field with signature (r_1, r_2) and torsion free unit rank $r = r_1 + r_2 - 1$,
S finite set of places of K not divisible by p,
ζ primitive pth root of unity,
$G := \mathrm{G}_{S,p}^{\infty}(K) = \mathrm{Gal}(\mathrm{F}_{S,p}^{\infty}(K)|K)$ the Galois group of the maximal pro-p extension $\mathrm{F}_{S,p}^{\infty}(K)$ of K which is unramified outside of S,
$d_1 := \dim_{\mathbb{F}_p} \mathrm{H}^1(G, \mathbb{F}_p)$ the *generator rank* of G,
$d_2 := \dim_{\mathbb{F}_p} \mathrm{H}^2(G, \mathbb{F}_p)$ the *relation rank* of G. Then

$$d_1 \leq d_2 \leq \begin{cases} d_1 + r & \text{if } S \neq \emptyset \text{ or } \zeta \notin K, \\ d_1 + 1 & \text{if } S = \emptyset \text{ and } \zeta \in K. \end{cases}$$

Corollary 4.1. $K = \mathbb{Q}(\sqrt{d})$ quadratic field with discriminant d and $S = \emptyset$,
$G := \mathrm{G}_p^{\infty}(K) = \mathrm{Gal}(\mathrm{F}_p^{\infty}(K)|K)$ the Galois group of the maximal unramified pro-p extension $\mathrm{F}_p^{\infty}(K)$ of K, i.e., the *p-class tower group* of K. Then

$$\begin{cases} d_2 = d_1 & \text{if } (d < 0 \text{ and } p \geq 3 \text{ odd}), \\ d_1 \leq d_2 \leq d_1 + 1 & \text{if either } (d < 0 \text{ and } p = 2) \text{ or } d > 0. \end{cases}$$

Remark. In the first case, G is called a Schur group, in the second case, it is a Schur+1 group.

[9] I. R. Shafarevich,
Extensions with prescribed ramification points,
Publ. Math., Inst. Hautes Études Sci. **18** (1963), 71–95 (Russian).
English transl. by J. W. S. Cassels:
Am.Math.Soc.Transl., II. Ser., **59** (1966), 128–149.

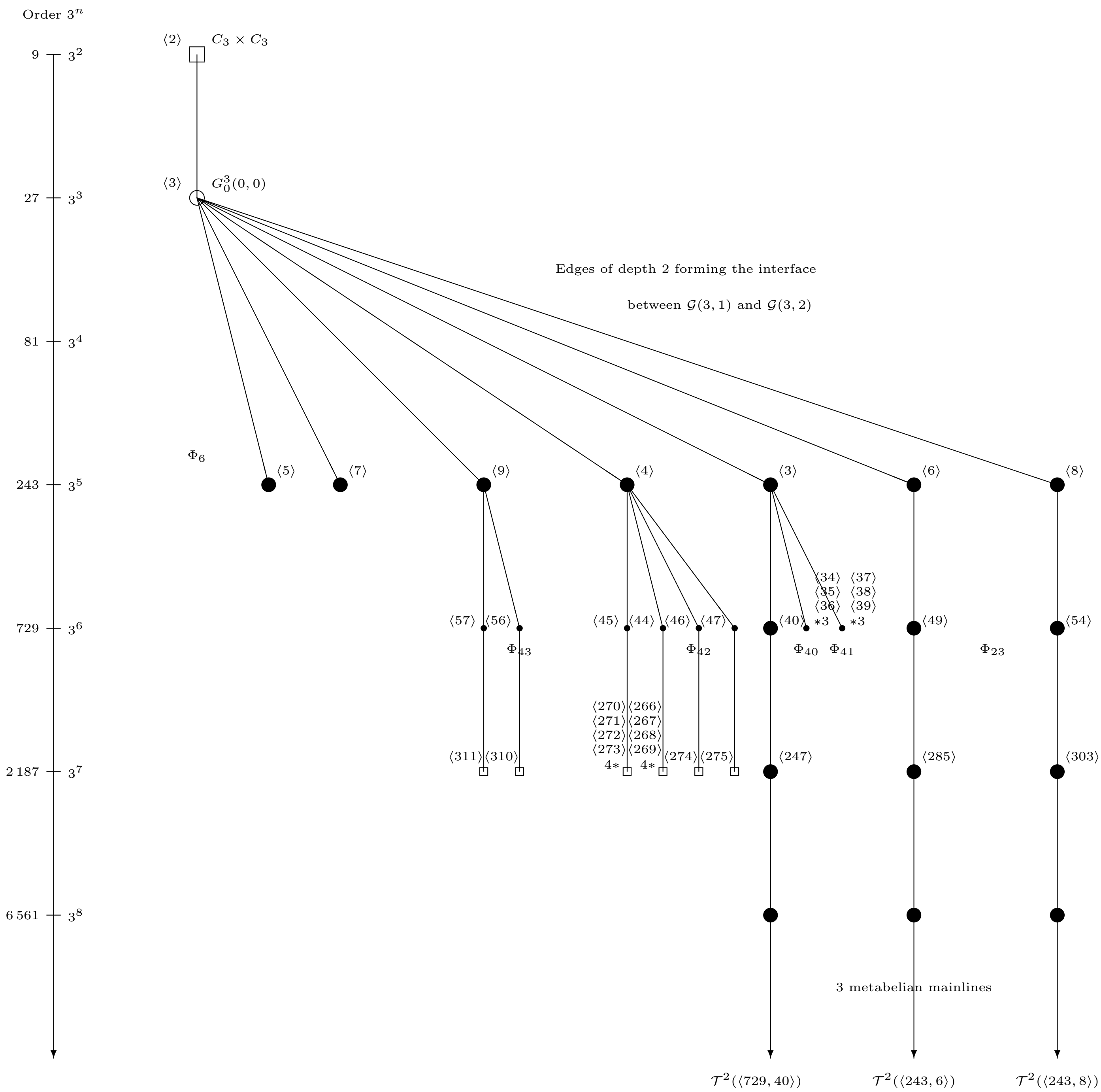
The tree of 3-groups G with G/G' of type $(3,3)$ in Figure 7 contains three basic kinds of vertices:

(1) the unique two metabelian Schur σ-groups $G = \langle 243, 5|7\rangle$ as **terminal vertices** (leaves) with $d_2 = d_1 = 2$, which are the 3-tower groups of fields with capitulation type D.5 or D.10,

(2) roots of **finite trees** with fixed coclass, such as $G = \langle 243, 4\rangle$, whose descendants are of TKT H.4 (Theorems 5.3, 7.1, 7.2),

(3) roots of **infinite coclass trees**, such as $G = \langle 243, 6|8\rangle$, which have all metabelian groups with TKTs in section E as descendants (Theorems 5.1, 5.2, 6.1).

FIGURE 7. Sporadic groups and tree roots G with $r = 2$, $c \geq 3$



§ 5. Three-Stage 3-Class Towers, identified by 3-Class Groups of Fields of Degree 18

Theorem 5.1. (Types E.8 and 9, Mayer, 2015) Let $p = 3$ and K a number field with 3-class group $\mathrm{Cl}_3(K)$ of type $(3, 3)$. Suppose $L_1, \ldots, L_4$ are the unramified cyclic cubic extensions of K within the first Hilbert 3-class field $\mathrm{F}_3^1(K)$. Let $\tau^{(1)}(L_i) = [\tau_0(L_i); \tau_1(L_i)]$ be the IPAD of L_i, for $1 \le i \le 4$.

If the 3-capitulation type $\varkappa_1(K)$ neither contains a total capitulation nor a 2-cycle, and K has the iterated IPAD of 2nd order

$$\tau^{(2)}(K) = [\tau_0(K); (\tau^{(1)}(L_1), \ldots, \tau^{(1)}(L_4))],$$

where $\tau_0(K) = 1^2$, $\tau^{(1)}(L_1) = [32; (2^2 1, (31^2)^3)]$, then

1.
$\tau^{(1)}(L_i) = [21; (2^2 1, (\mathbf{2}1)^3)]$, for $2 \le i \le 4$, $\Longrightarrow$ $\mathrm{G}_3^\infty(K) \simeq \langle 729, 54\rangle - \#\mathbf{1}; 2|4|6 = \langle 2187, 302|304|306\rangle$, one of three metabelian unbalanced σ-groups, and $\ell_3(K) = 2$,

2.
$\tau^{(1)}(L_i) = [21; (2^2 1, (\mathbf{3}1)^3)]$, for $2 \le i \le 4$, $\Longrightarrow$ $\mathrm{G}_3^\infty(K) \simeq \langle 729, 54\rangle - \#\mathbf{2}; 2|4|6$, one of three Schur σ-groups of derived length 3, and $\ell_3(K) = 3$,

The common component $2^2 1$ of all $\tau_1(L_i)$, $1 \le i \le 4$, is the type of the Hilbert 3-class field $\mathrm{F}_3^1(K)$.

Theorem 5.2. (Types E.6 and 14, Mayer, 2015) Let $p = 3$ and K a number field with 3-class group $\mathrm{Cl}_3(K)$ of type $(3, 3)$. Suppose $L_1, \ldots, L_4$ are the unramified cyclic cubic extensions of K within the first Hilbert 3-class field $\mathrm{F}_3^1(K)$. Let $\tau^{(1)}(L_i) = [\tau_0(L_i); \tau_1(L_i)]$ be the IPAD of L_i, for $1 \le i \le 4$.

If the 3-capitulation type $\varkappa_1(K)$
neither contains a total capitulation nor a 2-cycle, and K has the iterated IPAD of 2^{nd} order

$$\tau^{(2)}(K) = [\tau_0(K); (\tau^{(1)}(L_1), \ldots, \tau^{(1)}(L_4))],$$

where $\tau_0(K) = 1^2$, $\tau^{(1)}(L_1) = [32; (2^2 1, (31^2)^3)]$, then

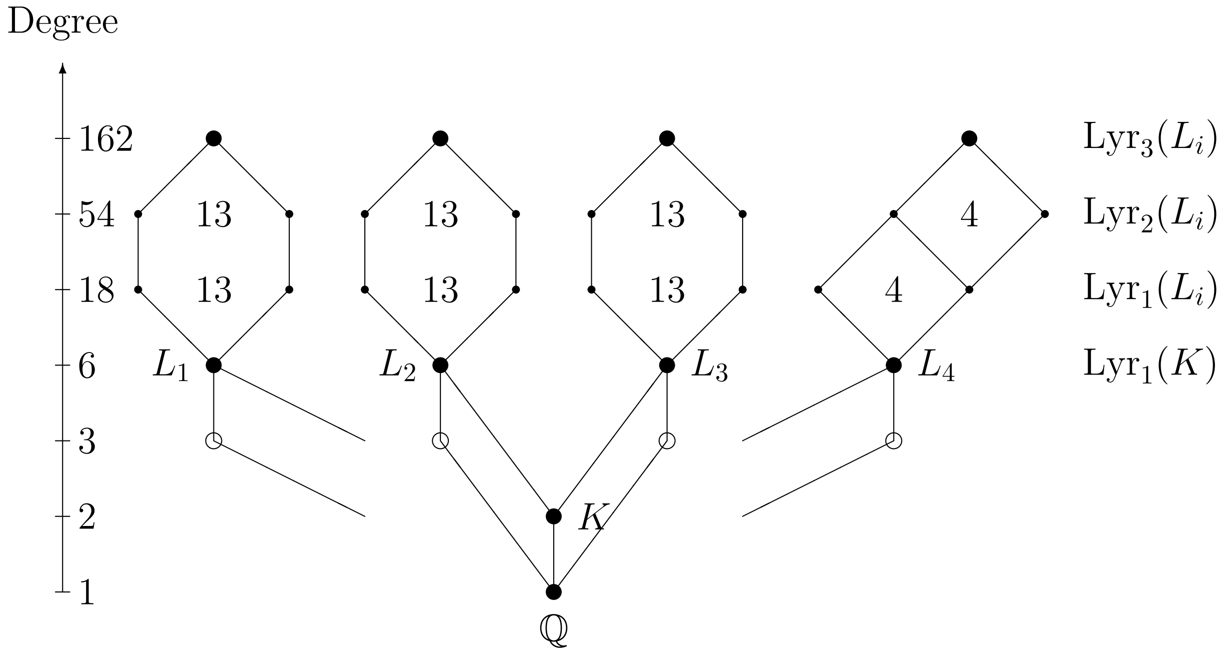
1.
$\tau^{(1)}(L_2) = [1^3; (2^2 1, (\mathbf{1}^3)^3, (1^2)^9)]$ and
$\tau^{(1)}(L_i) = [21; (2^2 1, (\mathbf{2}1)^3)]$, for $3 \le i \le 4$, $\Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 729, 49\rangle - \#\mathbf{1}; 4|5|6 = \langle 2187, 288|289|290\rangle$, one of three metabelian unbalanced σ-groups, and $\ell_3(K) = 2$,

2.
$\tau^{(1)}(L_2) = [1^3; (2^2 1, (\mathbf{2}1^2)^3, (1^2)^9)]$ and
$\tau^{(1)}(L_i) = [21; (2^2 1, (\mathbf{3}1)^3)]$, for $3 \le i \le 4$, $\Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 729, 49\rangle - \#\mathbf{2}; 4|5|6$, one of three Schur σ-groups of derived length 3, and $\ell_3(K) = 3$.

The common component $2^2 1$ of all $\tau_1(L_i)$, $1 \le i \le 4$, is the type of the Hilbert 3-class field $\mathrm{F}_3^1(K)$.

The various layers of extensions of a quadratic base field $K=\mathbb{Q}(\sqrt{d})$ with IPAD $\tau^{(1)}(K)=[1^2;((1^3)^3,21)]$ and capitulation type H.4 are shown in Figure 8.

FIGURE 8. Extension layers for the iterated IPAD $\tau_*^{(2)}(K)$ of second order



For proving Theorem 6.2 by means of Theorem 5.3, it suffices to compute the 3-class groups of fields of degree 18 in $\mathrm{Lyr}_1(L_i)$, $1\le i\le 4$. The proof of Theorem 7.2 with the aid of Theorem 7.1, however, requires the knowledge of 3-class groups of fields of degree 54 in $\mathrm{Lyr}_2(L_i)$, $1\le i\le 4$.

Theorem 5.3. (Type H.4, Part 1, Mayer, 2015) Let $p = 3$ and K a quadratic number field with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$. Suppose $L_1, \ldots, L_4$ are the unramified cyclic cubic extensions of K within the first Hilbert 3-class field $\mathrm{F}_3^1(K)$. Let $\tau^{(1)}(L_i) = [\tau_0(L_i); \tau_1(L_i)]$ be the IPAD of L_i, for $1 \le i \le 4$.

If K has the IPAD of 1$^{\mathrm{st}}$ order $\tau^{(1)}(K) = [1^2; ((1^3)^3, 21)]$, then the 3-capitulation type is $\varkappa_1(K) = (4111)$ and contains a 2-cycle.

If K has the iterated IPAD of 2$^{\mathrm{nd}}$ order

$$\tau^{(2)}(K) = [\tau_0(K); (\tau^{(1)}(L_1), \ldots, \tau^{(1)}(L_4))],$$

then

1.
$\tau^{(1)}(L_1) = [1^3; ((\mathbf{21^2})^\mathbf{4}, (1^2)^9)]$,
$\tau^{(1)}(L_i) = [1^3; (21^2, (21)^{12})]$, for $2 \le i \le 3$, and
$\tau^{(1)}(L_4) = [21; (21^2, (\mathbf{2}1)^3)] \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 2187, \mathbf{270}\rangle$, an unbalanced σ-group of derived length 3, and $\ell_3(K) = 3$,

2.
$\tau^{(1)}(L_1) = [1^3; (21^2, (\mathbf{1^3})^\mathbf{3}, (1^2)^9)]$,
$\tau^{(1)}(L_i) = [1^3; (21^2, (21)^{12})]$, for $2 \le i \le 3$, and
$\tau^{(1)}(L_4) = [21; (21^2, (\mathbf{3}1)^3)] \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 2187, \mathbf{271}|\mathbf{272}\rangle$, one of two unbalanced σ-groups of derived length 3, and $\ell_3(K) = 3$,

3.
$\tau^{(1)}(L_1) = [1^3; (21^2, (\mathbf{1^3})^\mathbf{3}, (1^2)^9)]$,
$\tau^{(1)}(L_2) = [1^3; (21^2, (21)^{12})]$,
$\tau^{(1)}(L_3) = [1^3; ((\mathbf{21^2})^\mathbf{4}, (\mathbf{2^2})^\mathbf{9})]$, and
$\tau^{(1)}(L_4) = [21; (21^2, (\mathbf{2}1)^3)] \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 2187, \mathbf{273}\rangle$, an unbalanced σ-group of derived length 3, and $\ell_3(K) = 3$.

In each of the three cases, $K = \mathbb{Q}(\sqrt{d})$ must be a real quadratic field with positive discriminant $d > 0$, since each of the 3-class tower groups $G = \mathrm{G}_3^\infty(K)$ is unbalanced with $d_2 = 3 > 2 = d_1$.

The common component 21^2 of all $\tau_1(L_i)$, $1 \le i \le 4$, is the type of the Hilbert 3-class field $\mathrm{F}_3^1(K)$.

§ 6. Application to Real Quadratic Fields

Theorem 6.1. (Section E, Mayer, 2015)
$K = \mathbb{Q}(\sqrt{d})$ real quadratic, discriminant $d > 0$.
In dependence on relevant values of $d < 10^7$, the length $\ell_3(K)$ and the group $\mathrm{G}_3^\infty(K)$ of the 3-class tower of K are given as follows:

$d \in \{342\,664,\ 1\,452\,185,\ 1\,787\,945,\ 4\,861\,720,\ 5\,976\,988,\ 8\,079\,101,\ 9\,674\,841\} \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 729, 54\rangle - \#2; 2|6$, $\ell_3(K) = 3$ (E.9).

$d \in \{4\,760\,877,\ 6\,652\,929,\ 7\,358\,937,\ 9\,129\,480\} \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 2187, 302|306\rangle$, $\ell_3(K) = 2$ (E.9).

$d \in \{6\,098\,360,\ 7\,100\,889\} \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 729, \mathbf{54}\rangle - \#\mathbf{2}; \mathbf{4}$, $\ell_3(K) = 3$ (E.8).

$d \in \{8\,632\,716\} \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 2187, \mathbf{304}\rangle$, $\ell_3(K) = 2$ (E.8).

$d \in \{3\,918\,837,\ 8\,897\,192,\ 9\,991\,432\} \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 2187, 289|290\rangle$, $\ell_3(K) = 2$ (E.14).

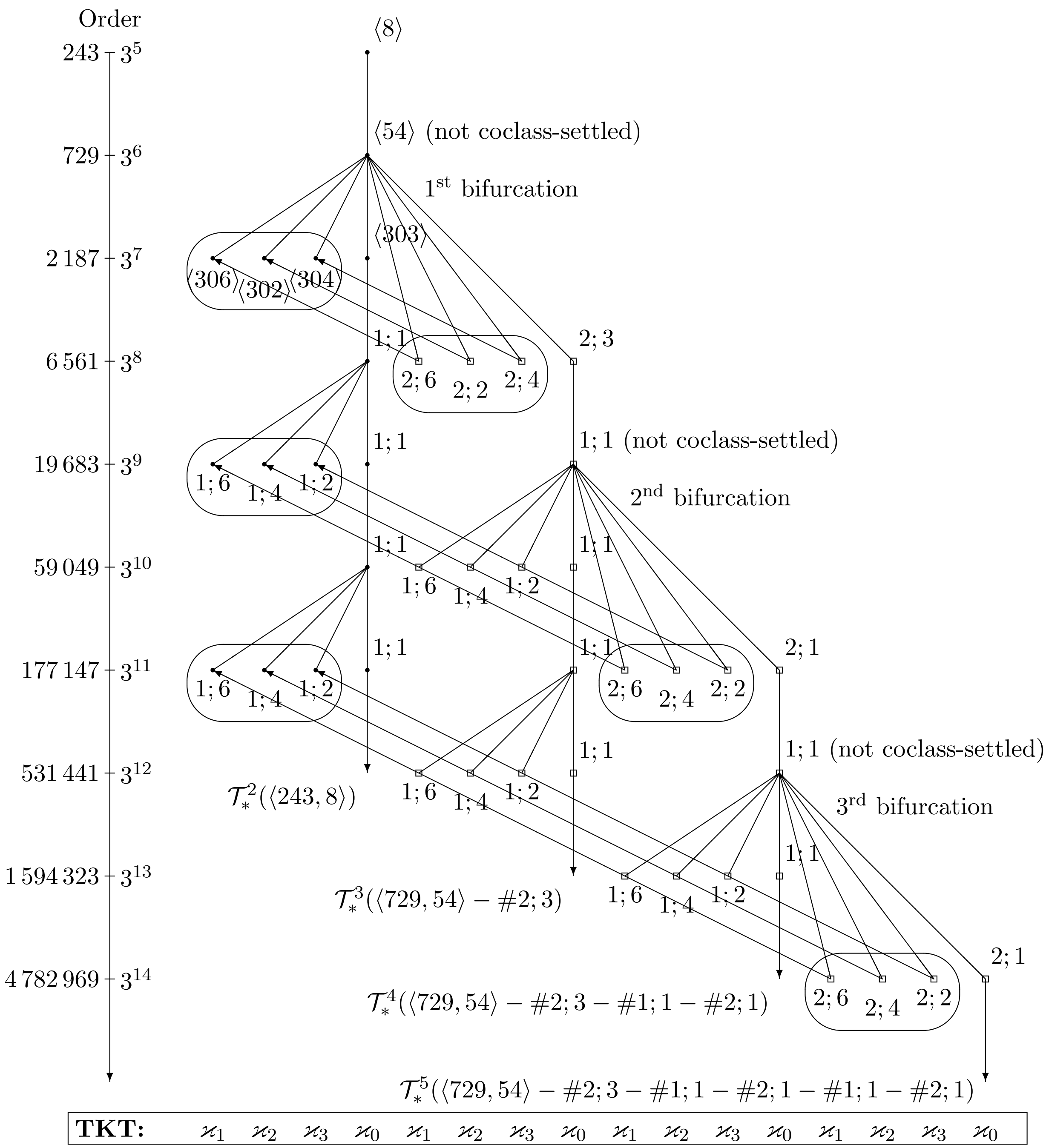
$d \in \{5\,264\,069,\ 6\,946\,573\} \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 729, \mathbf{49}\rangle - \#\mathbf{2}; \mathbf{4}$, $\ell_3(K) = 3$ (E.6).

$d \in \{7\,153\,097\} \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 2187, \mathbf{288}\rangle$, $\ell_3(K) = 2$ (E.6).

$d \in \{9\,433\,849\} \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 729, 49\rangle - \#2; 5|6$, $\ell_3(K) = 3$ (E.14).

The tree in Figure 9 contains the three non-metabelian Schur σ-groups $G = \langle 729, 54\rangle - \#2; 2|4|6$ with $d_2 = d_1 = 2$ and G/G' of type $(3, 3)$.

FIGURE 9. Projections from non-metabelian 3-tower groups G onto $\mathfrak{G} \in \mathcal{T}_*(\langle 243, 8\rangle)$



The tree in Figure 10 contains the three non-metabelian Schur σ-groups $G = \langle 729, 49\rangle - \#2; 4|5|6$ with $d_2 = d_1 = 2$ and G/G' of type $(3, 3)$.

FIGURE 10. Projections from non-metabelian 3-tower groups G onto $\mathfrak{G} \in \mathcal{T}_*(\langle 243, 6\rangle)$

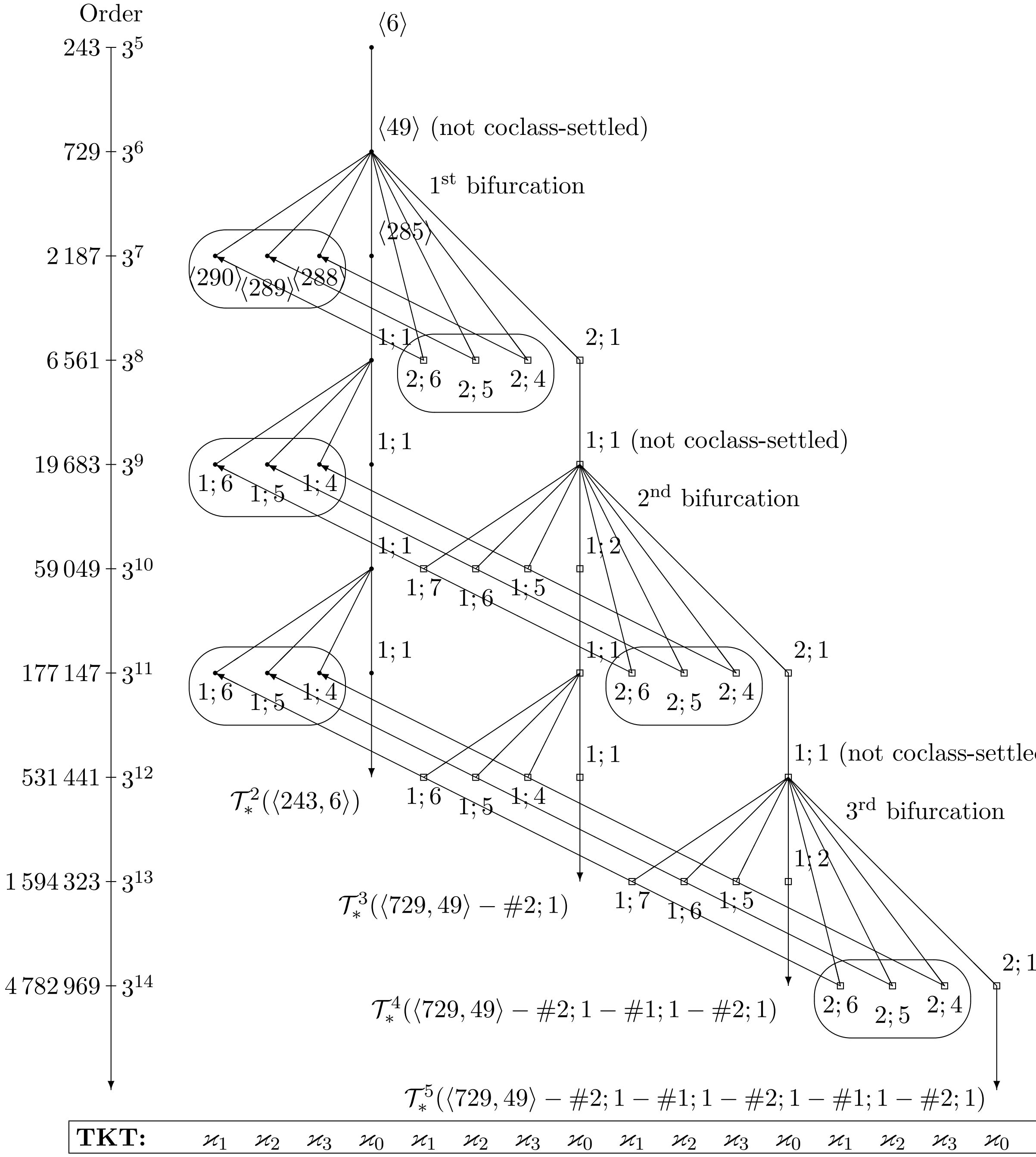


Figure 11 shows the population of the coclass tree with root $\langle 243, 8\rangle$.

FIGURE 11. Population of metabelian quotients $\mathfrak{G} = G/G'' \in \mathcal{T}^2(\langle 243, 8\rangle)$ of G

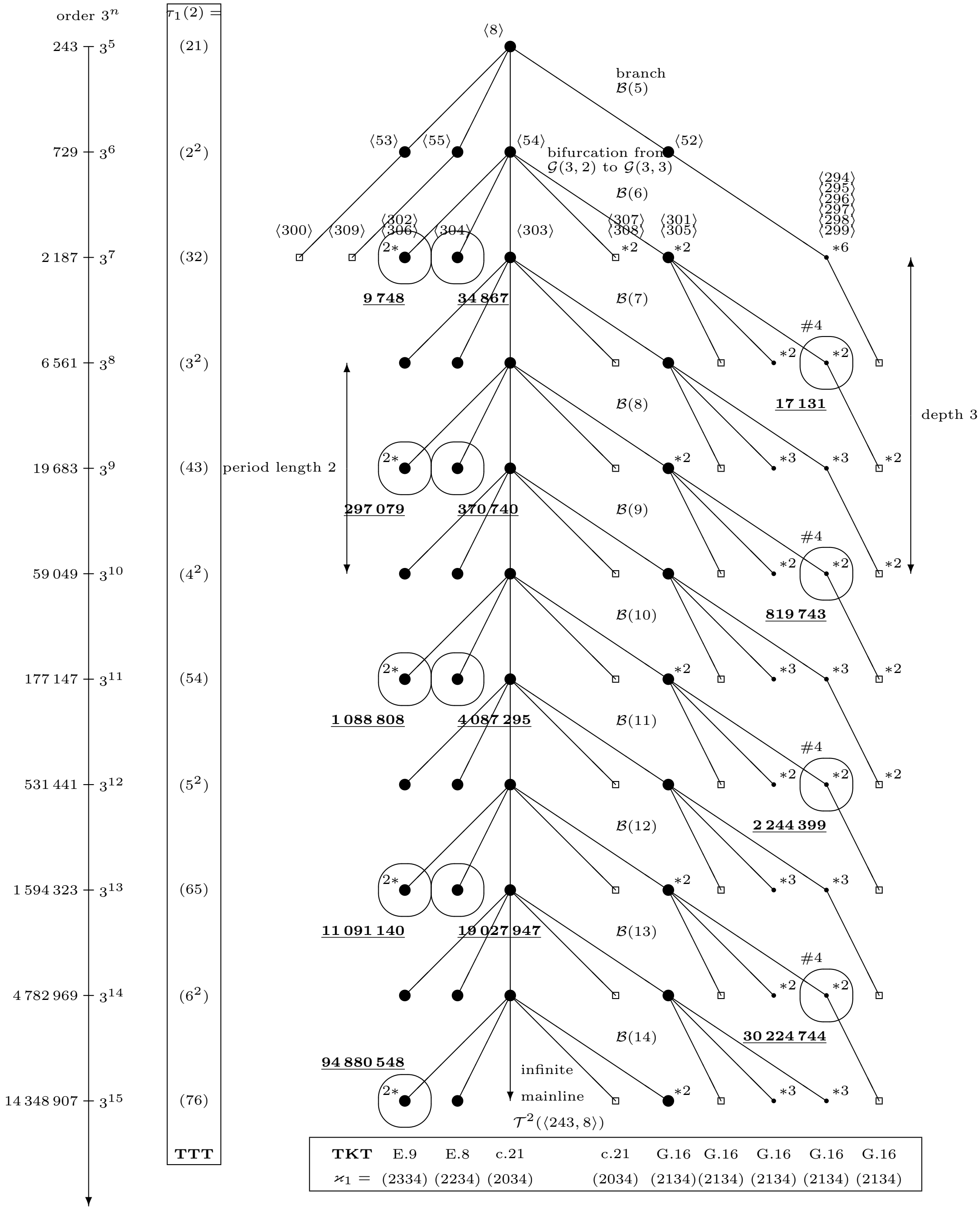
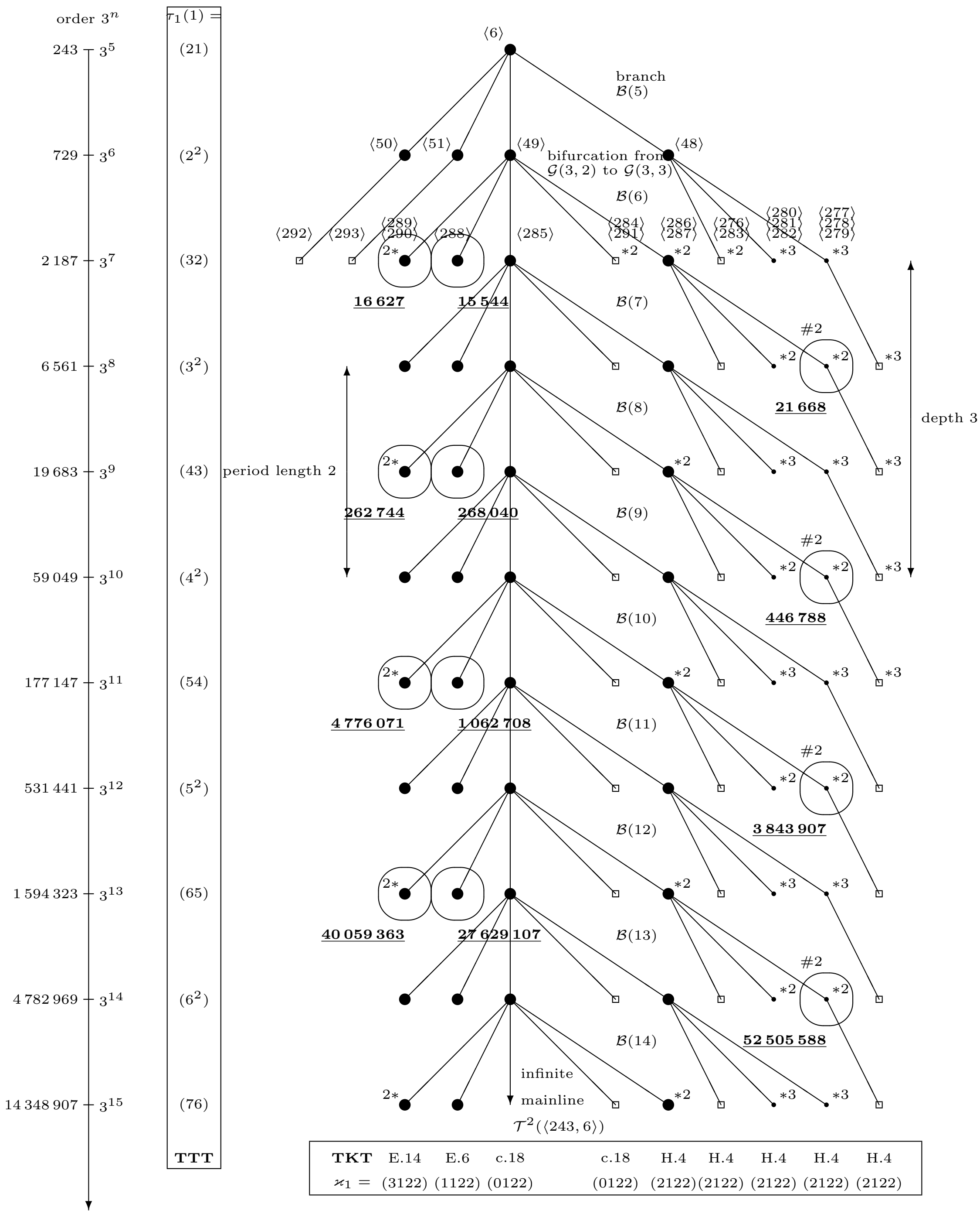


Figure 12 shows the population of the coclass tree with root $\langle 243, 6\rangle$.

FIGURE 12. Population of metabelian quotients $\mathfrak{G} = G/G'' \in \mathcal{T}^2(\langle 243, 6\rangle)$ of G



TKT	E.14	E.6	c.18	c.18	H.4	H.4	H.4	H.4	H.4
$\varkappa_1 =$	(3122)	(1122)	(0122)	(0122)	(2122)	(2122)	(2122)	(2122)	(2122)

Theorem 6.2. (Section H, Mayer, 2015)
$K = \mathbb{Q}(\sqrt{d})$ real quadratic, discriminant $d > 0$.
In dependence on relevant values of $d < 10^7$, the length $\ell_3(K)$ and the group $\mathrm{G}_3^\infty(K)$ of the 3-class tower of K are given as follows:

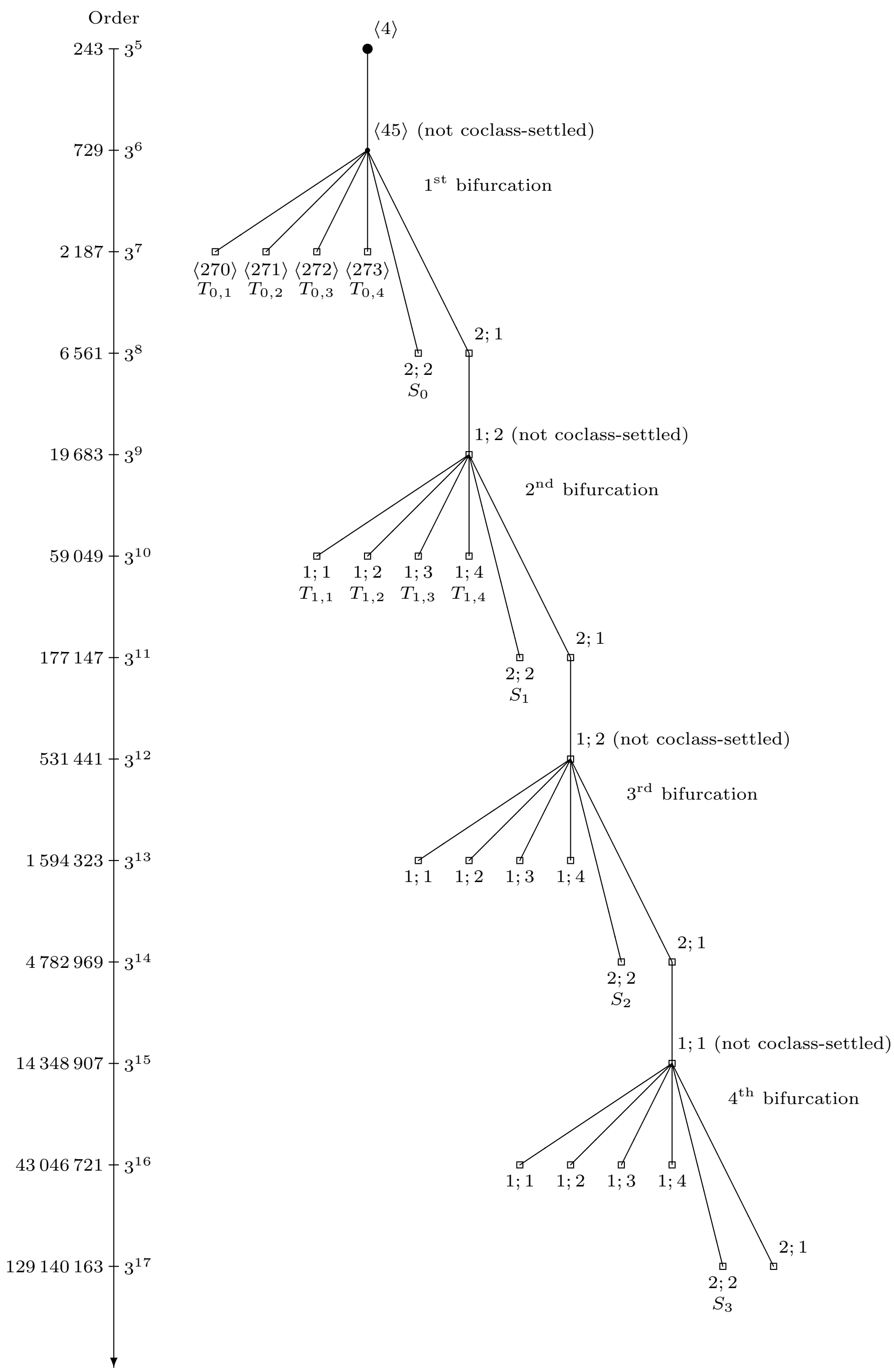
$d \in \{957\,013,\ 1\,571\,953,\ 1\,734\,184,$
$3\,517\,689,\ 4\,025\,909,\ 4\,785\,845,\ 4\,945\,973,$
$5\,562\,969,\ 6\,318\,733,\ 7\,762\,296,\ 8\,070\,637\} \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 2187, \mathbf{273}\rangle$, $\ell_3(K) = 3$.

$d \in \{2\,023\,845,\ 4\,425\,229,\ 6\,418\,369,\ 6\,469\,817$
$6\,775\,224,\ 6\,895\,612,\ 7\,123\,493,\ 9\,419\,261\} \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 2187, \mathbf{271}|\mathbf{272}\rangle$, $\ell_3(K) = 3$.

$d \in \{2\,303\,112,\ 3\,409\,817,\ 3\,856\,685,$
$5\,090\,485,\ 6\,526\,680\} \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 2187, \mathbf{270}\rangle$, $\ell_3(K) = 3$.

The tree in Figure 13 contains the non-metabelian Schur σ-group $G = \langle 729, 45\rangle - \#2;2$ with $d_2 = d_1 = 2$ and G/G' of type $(3,3)$.

FIGURE 13. Non-metabelian 3-tower groups $G \in \mathcal{T}_*(\langle 243, 4\rangle)$ with $\mathfrak{G} = \langle 729, 45\rangle$



§ 7. Three-Stage 3-Class Towers, identified by 3-Class Groups of Fields of Degree 54

Theorem 7.1. (Type H.4, Part 2, Mayer, 2015) Let $p = 3$ and K a quadratic number field with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$. Suppose $L_1, \ldots, L_4$ are the unramified cyclic cubic extensions of K within the first Hilbert 3-class field $\mathrm{F}_3^1(K)$. Let $\tau_*^{(1)}(L_i) = [\tau_0(L_i); \tau_1(L_i); \tau_2(L_i)]$ be the multi-layered IPAD of L_i, for $1 \le i \le 4$.

If K has the IPAD of 1^{st} order $\tau^{(1)}(K) = [1^2; ((1^3)^3, 21)]$ (and thus the 3-capitulation type $\varkappa_1(K) = (4111)$) and the iterated multi-layered IPAD of 2^{nd} order

$$\tau_*^{(2)}(K) = [\tau_0(K); (\tau_*^{(1)}(L_1), \ldots, \tau_*^{(1)}(L_4))],$$

then

1.
$\tau_*^{(1)}(L_1) = [1^3; ((\mathbf{21^2})^\mathbf{4}, (1^2)^9); (2^2 1, (\mathbf{2^2})^\mathbf{3}, (1^3)^3, (21)^6)]$,
$\tau_*^{(1)}(L_i) = [1^3; ((\mathbf{21^2})^\mathbf{4}, (\mathbf{2^2})^\mathbf{9}); (2^2 1, (\mathbf{21^2})^\mathbf{12})]$,
for $2 \le i \le 3$, and
$\tau_*^{(1)}(L_4) = [21; (21^2, (\mathbf{3}1)^3); (2^2 1, (\mathbf{2^2})^\mathbf{3})] \Longrightarrow$
either $\mathrm{G}_3^\infty(K) \simeq \langle 729, \mathbf{45}\rangle - \#\mathbf{2}; \mathbf{1}$, an unbalanced σ-group,
or $\mathrm{G}_3^\infty(K) \simeq \langle 729, \mathbf{45}\rangle - \#\mathbf{2}; \mathbf{2}$, a Schur σ-group,
in both cases, of order 3^8, derived length 3, and $\ell_3(K) = 3$,

2.
$\tau_*^{(1)}(L_1) = [1^3; ((\mathbf{21^2})^\mathbf{4}, (1^2)^9); (2^2 1, (\mathbf{32})^\mathbf{3}, (1^3)^3, (21)^6)]$,
$\tau_*^{(1)}(L_i) = [1^3; ((\mathbf{21^2})^\mathbf{4}, (\mathbf{2^2})^\mathbf{9}); (2^2 1, (\mathbf{31^2})^\mathbf{3}, (\mathbf{21^2})^\mathbf{9})]$,
resp.
$\tau_*^{(1)}(L_i) = [1^3; ((\mathbf{21^2})^\mathbf{4}, (\mathbf{2^2})^\mathbf{9}); ((\mathbf{2^2 1})^\mathbf{4}, (\mathbf{31^2})^\mathbf{9})]$,
for one or both $2 \le i \le 3$, and
$\tau_*^{(1)}(L_4) = [21; (21^2, (\mathbf{3}1)^3); (2^2 1, (\mathbf{32})^\mathbf{3})] \Longrightarrow$
either $G = \mathrm{G}_3^\infty(K)$, is an unbalanced σ-group of order $|G| \ge 3^9$, resp. $|G| \ge 3^{10}$,
or a Schur σ-group of order $|G| \ge 3^{11}$,
in both cases of derived length $\mathrm{dl}(G) \ge 3$, and $\ell_3(K) \ge 3$.

Theorem 7.2. (Tough Section H, Mayer, 2015)
$K = \mathbb{Q}(\sqrt{d})$ quadratic field with discriminant d.
In dependence on selected values of d, the length $\ell_3(K)$ and the group $\mathrm{G}_3^\infty(K)$ of the 3-class tower of K are given as follows:

$d \in \{-3\,896,\ -25\,447\} \Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 729, \mathbf{45}\rangle - \#\mathbf{2};\mathbf{2}$, $\ell_3(K) = 3$.

$d \in \{2\,852\,733,\ 8\,369\,468\} \Longrightarrow$
$\mathrm{G}_3^\infty(K)$ at least of order $\mathbf{3^8}$, $\ell_3(K) \geq 3$.

$d \in \{-6\,583,\ -23\,428, -27\,991\} \Longrightarrow$
$\mathrm{G}_3^\infty(K)$ at least of order $\mathbf{3^{11}}$, $\ell_3(K) \geq 3$.

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