# *5-Class towers of cyclic quartic fields arising from quintic reflection*

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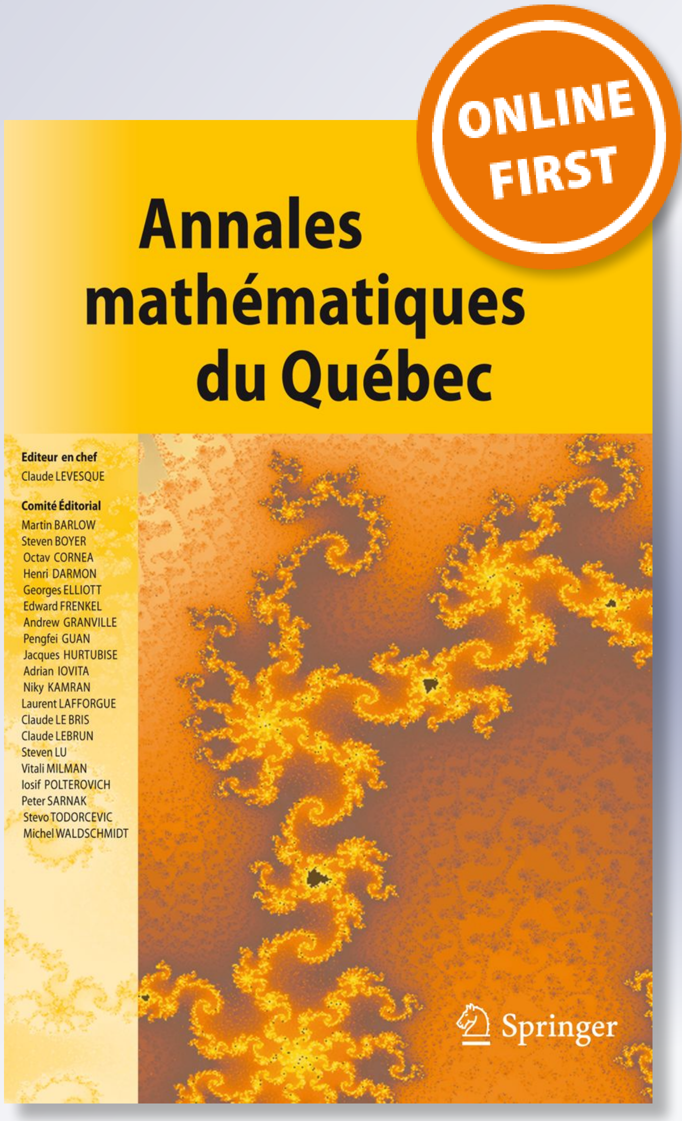


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# 5-Class towers of cyclic quartic fields arising from quintic reflection

**Abdelmalek Azizi**[1] · **Yasuhiro Kishi**[2] · **Daniel C. Mayer**[3] · **Mohamed Talbi**[4] · **Mohammed Talbi**[4]

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**Abstract**
Let $\zeta_5$ be a primitive fifth root of unity and $d \neq 1$ be a quadratic fundamental discriminant not divisible by 5. For the 5-dual cyclic quartic field $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$ of the quadratic fields $k_1 = \mathbb{Q}(\sqrt{d})$ and $k_2 = \mathbb{Q}(\sqrt{5d})$ in the sense of the quintic reflection theorem, the possibilities for the isomophism type of the Galois group $\mathrm{G}_5^{(2)}M = \mathrm{Gal}(M_5^{(2)}/M)$ of the second Hilbert 5-class field $M_5^{(2)}$ of $M$ are investigated, when the 5-class group $\mathrm{Cl}_5(M)$ is elementary bicyclic of rank two. Usually, the maximal unramified pro-5-extension $M_5^{(\infty)}$ of $M$ coincides with $M_5^{(2)}$ already. The precise length $\ell_5 M$ of the 5-class tower of $M$ is determined, when $\mathrm{G}_5^{(2)}M$ is of order less than or equal to $5^5$. Theoretical results are underpinned by the actual computation of all 83, respectively 93, cases in the range $0 < d < 10^4$, respectively $-2 \cdot 10^5 < d < 0$.

**Keywords** 5-Class field tower · 5-Principalization · Quadratic fields · 5-Dual cyclic quartic fields · Frobenius fields; finite 5-groups · Schur $\sigma$-groups

**Résumé**
Soient $\zeta_5$ une racine primitive 5-ième de l'unité et $d \neq 1$ un discriminant fondamental quadratique non divisible par 5. Pour le corps quartique cyclique $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$, le 5-dual des corps quadratiques $k_1 = \mathbb{Q}(\sqrt{d})$ et $k_2 = \mathbb{Q}(\sqrt{5d})$ au sens du théorème de réflexion, les possibilités pour le type d'isomorphisme du groupe de Galois $\mathrm{G}_5^{(2)}M = \mathrm{Gal}(M_5^{(2)}/M)$ du second 5-corps de classes de Hilbert $M_5^{(2)}$ de $M$ sont examinées lorsque le 5-groupe de classes $\mathrm{Cl}_5(M)$ est de type $(5, 5)$. En général, la pro-5-extension maximale non ramifiée $M_5^{(\infty)}$ de $M$ coïncide avec $M_5^{(2)}$. La longueur précise $\ell_5 M$ de la tour des 5-corps de classes de Hilbert de $M$ est déterminée lorsque $\mathrm{G}_5^{(2)}M$ est d'ordre inférieur ou égal à $5^5$. Les résultats théoriques

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sont étayés par le calcul réel de tous les 83 (resp. 93) cas dans l'intervalle $0 < d < 10^4$, (resp. $-2 \cdot 10^5 < d < 0$).

**Mathematics Subject Classification** Primary 11R37 · 11R29 · 11R11 · 11R16 · 11R20 · 11Y40; Secondary 20D15

## 1 Introduction

The present article arose from the desire to generalize our results [1] for the second 3-class group $\mathrm{Gal}(k_3^{(2)}/k)$ of the bicyclic biquadratic field $k = \mathbb{Q}(\sqrt{-3}, \sqrt{d})$, which is the compositum of 3-dual quadratic fields $k_1 = \mathbb{Q}(\sqrt{d})$ and $k_2 = \mathbb{Q}(\sqrt{-3d})$ in the cubic reflection theorem, to the situation of the quintic reflection theorem.

The precise statement of both reflection theorems requires the concept of *virtual units*. Let $p$ be a prime number and $K$ be a number field with multiplicative group $K^\times = K \backslash \{0\}$, maximal order $\mathcal{O}$, unit group $U$, fractional ideal group $\mathcal{I}$, and $p$-class rank $\varrho_p$. The quotient $V_p = I_p / \left(K^\times\right)^p$, where

$$I_p = \left\{\alpha \in K^\times \mid \alpha\mathcal{O} = \mathfrak{a}^p \text{ for some } \mathfrak{a} \in \mathcal{I}\right\},$$

is an elementary abelian $p$-group of rank $\sigma_p = \varrho_p + \dim_{\mathbb{F}_p}(U/U^p)$ and is called the *p-Selmer group* of non-trivial *p-virtual units*, that is, generators of principal $p$th powers of ideals of $K$. We refer to $\sigma_p$ as the *p-Selmer rank* of $K$.

### 1.1 Cubic reflection theorem

It is well known that the 3-Selmer ranks $\sigma_3(k_1)$ and $\sigma_3(k_2)$ of 3-dual quadratic fields $k_1 = \mathbb{Q}(\sqrt{d})$ and $k_2 = \mathbb{Q}(\sqrt{-3d})$ ($d > 0$ square-free) with respect to the quadratic cyclotomic mirror field $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, $\zeta_3 = \exp(2\pi i/3)$, satisfy the *cubic reflection theorem*

$$\sigma_3(k_2) = \sigma_3(k_1) - \delta, \tag{1.1}$$

which is a consequence of comparing the numbers of cyclic cubic extensions of $k_1$ and $k_2$ which are unramified outside 3 from the viewpoint of both, class field theory and Kummer theory. The invariant $0 \le \delta \le 1$ depends on the 3-virtual units of $k_1$ and $k_2$. More precisely, we have

$$\delta = \begin{cases} 0, & \text{if } V_3(k_2) \text{ (imaginary) contains a 3-virtual unit which is not 3-primary,} \\ 1, & \text{if } V_3(k_1) \text{ (real) contains a 3-virtual unit which is not 3-primary.} \end{cases} \tag{1.2}$$

### 1.2 Quintic reflection theorem

If $d \neq 1$ denotes a square-free integer prime to 5, then the 5-Selmer ranks $\sigma_5(k_1)$, $\sigma_5(k_2)$ of associated quadratic fields $k_1 = \mathbb{Q}(\sqrt{d})$, $k_2 = \mathbb{Q}(\sqrt{5d})$ and the 5-class rank $\varrho_5(M)$ of their 5-dual cyclic quartic field $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$, $\zeta_5 = \exp(2\pi i/5)$, with respect to the quartic cyclotomic mirror field $k_0 = \mathbb{Q}(\zeta_5)$ satisfy the *quintic reflection theorem*

$$\varrho_5(M) = \sigma_5(k_1) + \sigma_5(k_2) - \delta_1 - \delta_2, \tag{1.3}$$

where the invariants $0 \le \delta_1, \delta_2 \le 1$ depend on the 5-virtual units of $k_1$ and $k_2$ [9, p. 2]. The formula is derived by comparing the numbers of cyclic quintic extensions of $k_1$, $k_2$ and $M$

which are unramified outside of 5. The maximal real subfield of $k_0 = \mathbb{Q}(\zeta_5)$ is the quadratic field $k_0^+ = \mathbb{Q}(\sqrt{5})$.

### 1.3 Overview

The layout of this article is as follows. In Sect. 2, we prove that the action of the absolute Galois group $\mathrm{Gal}(M/\mathbb{Q})$ on the 5-class group $\mathrm{Cl}_5(M)$ considerably reduces the possibilities for the metabelianization $\mathrm{G}_5^{(2)}M$ of the 5-class tower group $\mathrm{G}_5^{(\infty)}M$ of $M$. In Sect. 3, it is shown that the six unramified cyclic quintic relative extensions $E_i/M$, $1 \le i \le 6$, give rise to absolute extensions $E_i/\mathbb{Q}$ which are either Frobenius or non-Galois. Using class number relations for the dihedral subextensions $E_i/k_0^+$ of $E_i/\mathbb{Q}$, we determine further constraints for the second 5-class group $\mathrm{G}_5^{(2)}M$, the 5-class tower group $\mathrm{G}_5^{(\infty)}M$, and the length $\ell_5 M$ of the 5-class tower in Sect. 4. The paper concludes with tables of concrete numerical realizations in Sect. 5 which underpin all theoretical statements and additionally reveal the statistical distribution of possible cases.

## 2 *p*-Principalization enforced by Galois action

The generating automorphism $\sigma$ of a cyclic number field $F/\mathbb{Q}$ of degree $d$ with Galois group $\mathrm{Gal}(F/\mathbb{Q}) = \langle\sigma\rangle$ acts on the class group $\mathrm{Cl}(F)$ of $F$ and thus also on the higher $p$-class groups $\mathrm{G}_p^{(n)}F$ with $n \in \mathbb{N} \cup \{\infty\}$, for a fixed prime number $p$. When $d$ and $p$ are coprime, a remarkable restriction of the possibilities for the metabelian second $p$-class group $\mathfrak{M} = \mathrm{G}_p^{(2)}F$ and consequently for the transfer kernel type $\varkappa(F)$ of $F$ is due to the fact that the trace $T_\sigma = \sum_{i=0}^{d-1}\sigma^i$ of $\sigma$ annihilates the commutator quotient of all the groups $\mathrm{G}_p^{(n)}F$.

**Definition 2.1** Let $p$ be a prime number and $G$ be a pro-$p$-group with finite abelianization $G/G'$. Suppose that $d \ge 2$ is a fixed integer. $G$ is said to be a $\sigma$-*group of degree* $d$, if $G$ possesses an automorphism $\sigma$ of order $d$ whose trace

$$T_\sigma = \sum_{j=0}^{d-1} \sigma^j \in \mathbb{Z}[\mathrm{Aut}(G)]$$

annihilates $G$ modulo $G'$, that is, if there exists $\sigma \in \mathrm{Aut}(G)$ such that $\mathrm{ord}(\sigma) = d$ and

$$x^{T_\sigma} = \prod_{j=0}^{d-1} \sigma^j(x) \in G'$$

for all $x \in G$.

We show that an epimorphism with characteristic kernel preserves the property of being a $\sigma$-group of degree $d$.

**Theorem 2.1** *Let $\phi : G \to H$ be an epimorphism of groups, whose kernel* $\ker(\phi)$ *is characteristic in $G$. If $G$ is a $\sigma$-group of degree $d$ coprime to $p$, then $H$ is also a $\sigma$-group of degree $d$.*

***Proof*** If $G$ is a $\sigma$-group of degree $d$, then there exists an automorphism $\sigma \in \mathrm{Aut}(G)$ of order $\mathrm{ord}(\sigma) = d$ such that

$$x^{T_\sigma} = \prod_{i=0}^{d-1} \sigma^i(x) \in G'$$

for all $x \in G$. According to [16, Th. 6.2], there exists an induced automorphism $\hat{\sigma} \in \mathrm{Aut}(H)$ such that $\hat{\sigma} \circ \phi = \phi \circ \sigma$. By induction we obtain $\hat{\sigma}^n \circ \phi = \phi \circ \sigma^n$, for all $n \in \mathbb{Z}$: let $n \geq 2$ be an integer and assume that $\hat{\sigma}^{n-1} \circ \phi = \phi \circ \sigma^{n-1}$, then

$$\hat{\sigma}^n \circ \phi = \hat{\sigma}^{n-1} \circ \hat{\sigma} \circ \phi = \hat{\sigma}^{n-1} \circ \phi \circ \sigma = \phi \circ \sigma^{n-1} \circ \sigma = \phi \circ \sigma^n.$$

Furthermore, $(\sigma^{-1})\hat{} = \hat{\sigma}^{-1}$. Now let $y \in H$. Since $\phi$ is surjective, there exists $x \in G$ with $\phi(x) = y$, and we obtain, as required,

$$\begin{aligned} y^{T_{\hat{\sigma}}} &= \prod_{i=0}^{d-1} \hat{\sigma}^i(y) = \prod_{i=0}^{d-1} \hat{\sigma}^i(\phi(x)) = \prod_{i=0}^{d-1} \phi(\sigma^i(x)) = \phi\left(\prod_{i=0}^{d-1} \sigma^i(x)\right) \\ &= \phi\left(x^{T_\sigma}\right) \in \phi(G') = H'. \end{aligned}$$

□

**Corollary 2.1** *In a descendant tree $\mathcal{T}$ of finite $p$-groups with edges $\pi : G \to \pi G$, the property of* ***not*** *being a $\sigma$-group of degree $d$ is inherited from the parent $\pi G$ by the immediate descendant $G$.*

***Proof*** The parent operator $\pi : G \to \pi G$ is the canonical projection from $G$ onto the quotient $\pi G = G/\gamma_c G$ by the last non-trivial member $\gamma_c G$, $c = \mathrm{cl}(G)$, of the lower central series $(\gamma_i G)_{i \geq 1}$ of $G$, and thus $\pi$ is an epimorphism with characteristic kernel $\ker(\pi) = \gamma_c G$, whence Theorem 2.1 justifies the claim. □

***Remark 2.1*** A $\sigma$-group $G$ in the classical sense is a $\sigma$-group of degree 2 in the new sense, since $x\sigma(x) \in G'$ is equivalent with $\sigma(x)G' = x^{-1}G'$. Such a group is also referred to as a group with *generator inverting* automorphism or briefly GI-automorphism.

**Theorem 2.2** (i) *The $p$-class tower group $\mathrm{G}_p^{(\infty)}F$ and all higher $p$-class groups $\mathrm{G}_p^{(n)}F$ with $n \geq 2$ of a cyclic quartic number field $F$ are $\sigma$-groups of degree* 4.

(ii) *When the quadratic subfield $k < F$ has a trivial $p$-class group, the groups $\mathrm{G}_p^{(\infty)}F$ and $\mathrm{G}_p^{(n)}F$ with $n \geq 2$ are simultaneously $\sigma$-groups of degree* 2.

***Proof*** (i) The generating automorphism $\sigma$ of $F/\mathbb{Q}$ annihilates the class group $\mathrm{Cl}(F)$ when it acts by its trace $T_\sigma = \sum_{i=0}^{3} \sigma^i \in \mathbb{Z}[\langle\sigma\rangle]$, since

$$x^{T_\sigma} = \prod_{i=0}^{3} \sigma^i(x) = \mathrm{N}_{F/\mathbb{Q}}(x) \in \mathrm{Cl}(\mathbb{Q}) = 1$$

for all $x \in \mathrm{Cl}(F)$. Of course, the same is true for all $p$-class groups $\mathrm{Cl}_p(F)$ with primes $p$. Finally, for any $n \in \mathbb{N} \cup \{\infty\}$, we have the isomorphisms

$$\mathrm{G}_p^{(n)}F / \left(\mathrm{G}_p^{(n)}F\right)' \simeq \mathrm{Cl}_p(F).$$

(ii) When the unique (real) quadratic subfield $k < F$ has trivial $p$-class group $\mathrm{Cl}_p(k) = 1$, the relative automorphism $\tau = \sigma^2 \in \mathrm{Gal}(F/k)$ with order 2 acts by inversion on $\mathrm{Cl}_p(F)$, since

$$x^{T_\tau} = x^{1+\tau} = x \cdot \tau(x) = \mathrm{N}_{F/k}(x) \in \mathrm{Cl}_p(k) = 1,$$

and thus $x^\tau = x^{-1}$ for all $x \in \mathrm{Cl}_p(F)$.

□

**Table 1** The Artin pattern of the twelve 5-groups of order $5^5$ in the stem of $\Phi_6$

| Identifier of the 5-group | | Flag | 5-Principalization type | | |
|---|---|---|---|---|---|
| James | Small group | $f$ | $\varkappa$ | Cycle pattern | Property |
| $\Phi_6(2^2 1)_a$ | $\langle 3125, 14\rangle *$ | 1 | (123456) | (1)(2)(3)(4)(5)(6) | Identity |
| $\Phi_6(2^2 1)_{b_1}$ | $\langle 3125, 11\rangle *$ | 1 | (125364) | (1)(2)(3564) | 4-Cycle |
| $\Phi_6(2^2 1)_{b_2}$ | $\langle 3125, 7\rangle$ | 1 | (126543) | (1)(2)(36)(45) | Two 2-Cycles |
| $\Phi_6(2^2 1)_{c_1}$ | $\langle 3125, 8\rangle *$ | 0 | (612435) | (16532)(4) | 5-Cycle |
| $\Phi_6(2^2 1)_{c_2}$ | $\langle 3125, 13\rangle *$ | 0 | (612435) | (16532)(4) | 5-Cycle |
| $\Phi_6(2^2 1)_{d_0}$ | $\langle 3125, 10\rangle$ | 0 | (214365) | (12)(34)(56) | Three 2-Cycles |
| $\Phi_6(2^2 1)_{d_1}$ | $\langle 3125, 12\rangle *$ | 0 | (512643) | (154632) | 6-Cycle |
| $\Phi_6(2^2 1)_{d_2}$ | $\langle 3125, 9\rangle *$ | 0 | (312564) | (132)(456) | Two 3-Cycles |
| $\Phi_6(21^3)_a$ | $\langle 3125, 4\rangle$ | 1 | (022222) | | nrl. const. with fp. |
| $\Phi_6(21^3)_{b_1}$ | $\langle 3125, 5\rangle$ | 1 | (011111) | | Nearly constant |
| $\Phi_6(21^3)_{b_2}$ | $\langle 3125, 6\rangle$ | 1 | (011111) | | Nearly constant |
| $\Phi_6(1^5)$ | $\langle 3125, 3\rangle$ | 1 | (000000) | | Constant |

***Remark 2.2*** A pro-$p$-group $G$ with finite abelianization $G/G'$ is called a *strong* $\sigma$-group if it possesses an automorphism $\sigma$ of order 2 which acts as inversion on both cohomology groups $\mathrm{H}^1(G, \mathbb{F}_p)$ and $\mathrm{H}^2(G, \mathbb{F}_p)$. We emphasize the following two facts:

- An epimorphism does not necessarily preserve the property of being a strong $\sigma$-group.
- Whereas the group $\mathrm{G}_p^{(\infty)} F$ of a quadratic field $F$ is a strong $\sigma$-group, according to Schoof [19, Lem. 4.1, p. 217], this is not necessarily the case for a cyclic quartic field $F$. See for instance the unusual cases in Theorem 4.5.

In view of our special situation with $p = 5$, $F = M$, $\mathrm{Cl}_5(M) = (5, 5)$ and $k = k_0^+$, we tested finite metabelian 5-groups $G$ with $G/G' \simeq (5, 5)$ of order $|G| = 3125 = 5^5$ and coclass $\mathrm{cc}(G) = 2$, for the property of simultaneously being a $\sigma$-group of degree 4 and degree 2. These groups are crucial contestants for second 5-class groups $\mathrm{G}_5^{(2)} M$ and form the stem of Hall's isoclinism family $\Phi_6$. (See [13, §3.5, pp. 445–448] and [17, Sect.7, pp. 93–98].) In Table 1, the groups are characterized by their identifiers according to James [8] and the SmallGroups Library [2]. An asterisk $*$ marks a Schur $\sigma$-group, and a flag $f \in \{0, 1\}$ indicates a $\sigma$-group of simultaneous degrees 4 and 2.

**Theorem 2.3** *A finite 5-group $G$ with $G/G' \simeq (5, 5)$ which is a $\sigma$-group of degree 4 is either of coclass $\mathrm{cc}(G) = 1$ or isomorphic to one of the two Schur $\sigma$-groups $\langle 3125, i\rangle$ with $i \in \{11, 14\}$ or isomorphic to a descendant of one of the capable groups $\langle 3125, i\rangle$ with $i \in \{3, 4, 5, 6, 7\}$.*

***Proof*** Using permutation representations, we compiled a program script in Magma [12] for testing whether an assigned 5-group $G$ with $G/G' \simeq (5, 5)$ is a $\sigma$-group of degree 4. □

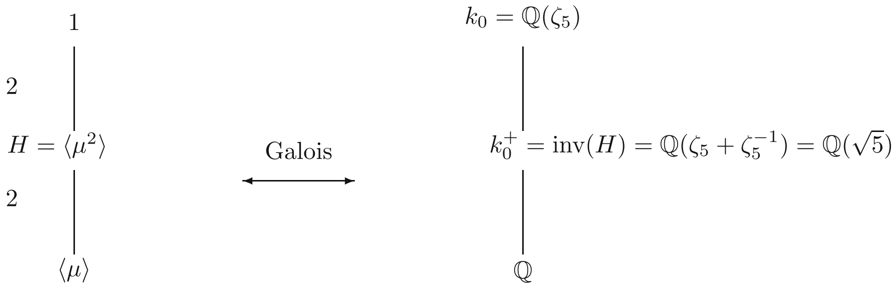


**Fig. 1** Galois correspondence between $\mathrm{Gal}(k_0/\mathbb{Q})$and $k_0$

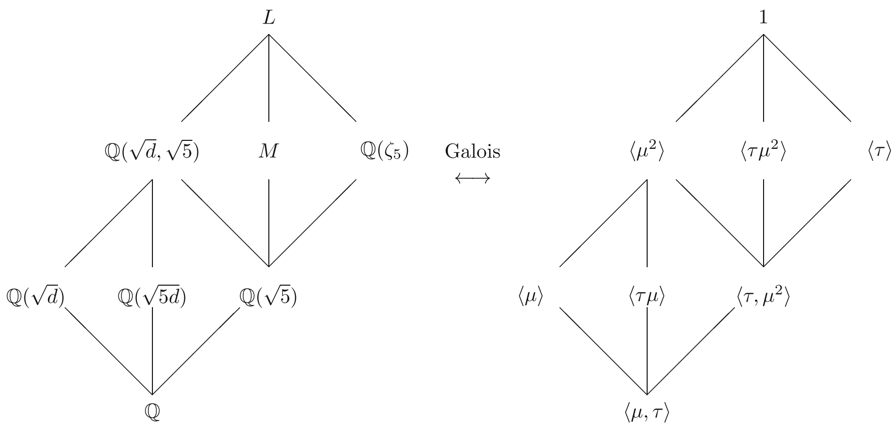


**Fig. 2** Galois correspondence between $L$and $\mathrm{Gal}(L/\mathbb{Q})$

# 3 Frobenius and non-Galois unramified 5-extensions

## 3.1 On the cyclic quartic fields *M*

Let $\zeta_5$ be a primitive 5th root of unity, then the irreducible polynomial of $\zeta_5$ is given by $\mathrm{Irr}_{\mathbb{Q}}(\zeta_5) = X^4 + X^3 + X^2 + X + 1$, and $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) = \langle \mu \rangle$ is a cyclic group of order 4 which admits one subgroup $\langle \mu^2 \rangle$ of order 2. By Galois correspondence, this subgroup corresponds to $\mathbb{Q}(\zeta_5)^+ = \mathbb{Q}(\zeta_5 + \zeta_5^{-1}) = \mathbb{Q}(\sqrt{5})$. (See Fig. 1.)

Let $d$ be a square-free integer prime to 5. Then $L = \mathbb{Q}(\sqrt{d}, \zeta_5)$ is a normal extension over $\mathbb{Q}$ of degree 8, and the Galois group is

$$\mathrm{Gal}\,(L/\mathbb{Q}) = \langle \tau, \mu \rangle = \{1, \tau, \mu, \mu^2, \mu^3, \tau\mu, \tau\mu^2, \tau\mu^3\}, \text{ where } \tau(\sqrt{d}) = -\sqrt{d}.$$

This is an abelian group of type (2, 4) which has six proper subgroups ordered as follows:

$$H_1 = \langle \tau \rangle,\ H_2 = \langle \mu \rangle,\ H_3 = \langle \mu^2 \rangle,\ H_4 = \langle \tau\mu \rangle,\ H_5 = \langle \tau\mu^2 \rangle \text{ and } H_6 = \langle \tau, \mu^2 \rangle.$$

Note that the subgroups $H_1,\ H_3,\ H_5$ are cyclic of order 2, the subgroups $H_2,\ H_4$ are cyclic of order 4, and the group $H_6$ is bicyclic of order 4. (See Fig. 2.)

We consider the field $M$ fixed by the subgroup $\langle \tau\mu^2 \rangle$. Then $M$ is a cyclic quartic field and can be generated by adjunction $M = \mathbb{Q}(\alpha)$ of

$$\alpha = (\zeta_5 - \zeta_5^{-1})\sqrt{d} = \sqrt{-\frac{5d}{2} - \frac{d}{2}\sqrt{5}}$$

to $\mathbb{Q}$. With respect to the quartic cyclotomic mirror field $k_0 = \mathbb{Q}(\zeta_5)$, $M$ satisfies the *quintic reflection theorem* (Eq. (1.3)).

**Lemma 3.1** (i) *Let $K$ be a number field and $F/K$ be a cyclic quartic extension. Then there exist $n$, $e$ and $f \neq 0$ in $K$ such that*

(1) *$n$ is not a square in $K$,*
(2) *$n(e^2 - f^2 n)$ is a square in $K$,*
(3) *$F = K(\alpha)$, where $\alpha = \sqrt{e + f\sqrt{n}}$,*

*and the minimal polynomial of $\alpha$ over $K$ is given by* $\mathrm{Irr}_K(\alpha) = X^4 - 2eX^2 + (e^2 - f^2 n)$.

(ii) *Conversely, if there exist numbers $n$, $e$ and $f \neq 0$ in $K$ which satisfy the conditions* (1), (2) *and* (3), *then $F/K$ is a cyclic quartic extension and the polynomial $P(X) = X^4 - 2eX^2 + (e^2 - f^2 n)$ is irreducible over $K$. In fact, $F$ is the splitting field of $P(X)$.*

***Proof*** (i) It is known that the group $\mathbb{Z}/4\mathbb{Z}$ has a single subgroup of order 2. By Galois theory, there exists a corresponding intermediary field $R$ of the cyclic quartic extension $F/K$. Thus we can find $n \in K$, which is not a square in $K$, such that $R = K(\sqrt{n})$. Since $F$ is a quadratic extension of $R$, there exists $\alpha \in F$ such that $\alpha^2 = e + f\sqrt{n} \in R$ with $e, f \in K$, $f \neq 0$, and $F = R(\alpha)$. Thus we have $K(\alpha) = F$, because $\alpha \notin R$. Furthermore, it is obvious that the minimal polynomial of $\alpha$ is $P(X) = X^4 - 2eX^2 + (e^2 - f^2 n)$ and the splitting field of $P(X)$ over $\mathbb{Q}$ is $F$.

The discriminant of $P$ is given by

$$D = 16(e^2 - f^2 n)(2f\sqrt{n})^4 = 2^8 f^4 n^2 (e^2 - f^2 n).$$

Therefore the Galois group $\mathrm{Gal}(F/K)$ can be seen as a subgroup of the permutation group of the roots of $P(X)$, which is isomorphic to $S_4$, and cannot be injected into $\mathcal{A}_4$, since the group $\mathcal{A}_4$ does not have a cyclic subgroup of order 4. We conclude that the discriminant is not a square in $K$, whence $K\left(\sqrt{e^2 - f^2 n}\right)/K$ is of degree 2 and is contained in $F$. It follows that

$$R = K\left(\sqrt{n}\right) = K\left(\sqrt{e^2 - f^2 n}\right),$$

so $\frac{e^2 - f^2 n}{n}$ is a square in $K$. Consequently, we see that $n\left(e^2 - f^2 n\right) = n^2 \frac{e^2 - f^2 n}{n}$ is a square in $K$.

(ii) Conversely, let

$$P(X) = X^4 - 2eX^2 + (e^2 - f^2 n)$$

with $n, e, f \in K$, $f \neq 0$, such that the conditions (1), (2) and (3) are satisfied. Since $\alpha$ is a root of $P(X)$, the degree $[F : K]$ must be a divisor of 4. Since $K\left(\sqrt{n}\right) \subseteq F$, we have either $F = K\left(\sqrt{n}\right)$ or $[F : K] = 4$. If we have $F = K\left(\sqrt{n}\right)$, there exist $u, v \in K$ such that $\sqrt{e + f\sqrt{n}} = u + v\sqrt{n}$. Thus

$$e^2 - f^2 n = \left(u^2 - v^2 n\right)^2,$$

and by (2) we conclude that $n$ is a square in $K$, which is a contradiction. So $[F : K] = 4$, and this enforces that $P(X)$ is the minimal polynomial of $\alpha$. From the fact

$$e - f\sqrt{n} = \left(\frac{1}{\sqrt{n}}\right)^2 \frac{n(e^2 - f^2 n)}{\alpha^2},$$

we conclude that $F$ is the splitting field of $P(X)$ over $K$. Moreover, $F$ is normal and #Gal $(F/K) = 4$. Now we prove that Gal $(F/K)$ is cyclic of order 4. If the Galois group Gal $(F/K)$ were isomorphic to $V_4$, then the discriminant would be a square in $K$. This would imply that $n$ were a square in $K$, which is a contradiction.

□

**Corollary 3.1** *Let $d$ be a square-free integer prime to 5 and $\zeta_5$ be a primitive 5th root of unity. Then the mirror image $M$ of $k_1 = \mathbb{Q}(\sqrt{d})$ can always be generated by adjoining the algebraic number*

$$\alpha = (\zeta_5 - \zeta_5^{-1})\sqrt{d} = \sqrt{\frac{-5d}{2} + \frac{-d}{2}\sqrt{5}}$$

*to the rational, whence $M$ is complex for $d > 0$ and $M$ is real for $d < 0$. For the construction of $M$ one can therefore use the minimal polynomial of $\alpha$ over $\mathbb{Q}$, which is given by*

$$\mathrm{Irr}_{\mathbb{Q}}(\alpha) = X^4 + 5dX^2 + 5d^2. \tag{3.1}$$

***Remark 3.1*** The conductor $c(M)$ and the discriminant $d(M)$ of $M$ are given by

$$c(M) = \begin{cases} 20d & \text{if } d \equiv 2, 3 \pmod 4, \\ 5d & \text{if } d \equiv 1 \pmod 4, \end{cases} \tag{3.2}$$

$$d(M) = c(M)^2 d(k_0^+) = \begin{cases} 2000d^2 & \text{if } d \equiv 2, 3 \pmod 4, \\ 125d^2 & \text{if } d \equiv 1 \pmod 4, \end{cases} \tag{3.3}$$

where $d(k_0^+) = 5$ is the discriminant of the quadratic subfield $k_0^+ = \mathbb{Q}(\sqrt{5})$ of $M$. (See [5,21].)

### 3.2 Imaginary cyclic quartic fields *M* with *d* > 0

In the following, the two Frobenius groups $F_{5,w}$ of order 20 with primitive root $w \in \{2, 3\}$ modulo 5 will be denoted by

$$\begin{cases} F_{5,2} = \langle\, \sigma, \iota \mid \sigma^5 = 1,\ \iota^4 = 1, \iota^{-1}\sigma\iota = \sigma^2 \,\rangle, \\ F_{5,3} = \langle\, \sigma, \iota \mid \sigma^5 = 1,\ \iota^4 = 1, \iota^{-1}\sigma\iota = \sigma^3 \,\rangle, \end{cases} \tag{3.4}$$

where $\iota|_M = \mu|_M$.

**Proposition 3.1** *Let $E_1, \ldots, E_6$ be the six unramified cyclic quintic extensions of the imaginary cyclic quartic field $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$, $d > 0$, with 5-class group $\mathrm{Cl}_5(M) \simeq C_5 \times C_5$ or, more generally, of 5-class rank 2. The properties of these fields as absolute extensions $E_i/\mathbb{Q}$, in dependence on the eight cases in Table 2, are given as follows:*

(1) *In cases* (a) *and* (g), *all six fields $E_1, \ldots, E_6$ are normal and share isomorphic automorphism groups* $\mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,2}$ *for $i = 1, \ldots, 6$.*

**Table 2** All possible 5-class ranks $r_1 := \varrho_5(k_1)$, $r_2 := \varrho_5(k_2)$ and invariants $\delta_1, \delta_2$ for the associated quadratic fields $k_1, k_2$ which are 5-dual to an imaginary cyclic quartic field $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$, $d > 0$, with 5-class rank $r := \varrho_5(M) = 2$

| Case | $r_1$ | $\delta_1$ | $r_2$ | $\delta_2$ |
|---|---|---|---|---|
| (a) | 1 | 0 | 0 | 1 |
| (b) | 0 | 1 | 1 | 0 |
| (c) | 1 | 1 | 1 | 1 |
| (d) | 0 | 0 | 0 | 0 |
| (e) | 1 | 1 | 0 | 0 |
| (f) | 0 | 0 | 1 | 1 |
| (g) | 2 | 1 | 0 | 1 |
| (h) | 0 | 1 | 2 | 1 |

(2) *In cases* (b) *and* (h)*, all six fields* $E_1, \ldots, E_6$ *are normal and share isomorphic automorphism groups* $\mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,3}$ *for* $i = 1, \ldots, 6$.

(3) *In all the other cases* (c), (d), (e), (f)*, two extensions are normal with non-isomorphic automorphism groups, say*

$$\mathrm{Gal}(E_1/\mathbb{Q}) \simeq F_{5,2} \quad \textit{and} \quad \mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3},$$

*but the other four extensions are non-Galois and form two conjugate pairs* $E_3 \simeq E_4$ *and* $E_5 \simeq E_6$.

***Proof*** According to the quintic reflection theorem [9], the assumption $r = 2$ implies that one of the eight disjoint cases in Table 2 is satisfied.

In case (a), the 5-Selmer group of $k_1$ is given by $V_5(k_1) = \langle \alpha_{11}, \varepsilon_1 \rangle$. See [9, p. 2, l. 3]. Let

$$E_1 := \mathrm{Spl}_{\mathbb{Q}} f(X, \alpha_{11}) \quad \text{and} \quad E_2 := \mathrm{Spl}_{\mathbb{Q}} f(X, \varepsilon_1).$$

In virtue of $\delta_1 = 0$, $E_1$ and $E_2$ are unramified cyclic quintic extensions of $M$. According to [9, p. 17, l. 9–23], $\mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,2}$, for $1 \le i \le 2$. Let $L := E_1 \cdot E_2$ be the compositum. Then, by [9, Lem. 2.5], all proper subextensions $E$ of $L/M$ have $\mathrm{Gal}(E/\mathbb{Q}) \simeq F_{5,2}$.

In case (b), the 5-Selmer group of $k_2$ is given by $V_5(k_2) = \langle \alpha_{21}, \varepsilon_2 \rangle$. See [9, p. 2, l. 3]. Let $E_1 := \mathrm{Spl}_{\mathbb{Q}} f(X, \alpha_{21})$ and $E_2 := \mathrm{Spl}_{\mathbb{Q}} f(X, \varepsilon_2)$. In virtue of $\delta_2 = 0$, $E_1$ and $E_2$ are unramified cyclic quintic extensions of $M$. According to [9, p. 17, l. 9–23], $\mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,3}$, for $1 \le i \le 2$. Let $L := E_1 \cdot E_2$ be the compositum. Then, by [9, Lem. 2.5], all proper subextensions $E$ of $L/M$ have $\mathrm{Gal}(E/\mathbb{Q}) \simeq F_{5,3}$.

Exemplarily, we consider case (d). Then the 5-Selmer groups of $k_1$ and $k_2$ are given by $V_5(k_1) = \langle \varepsilon_1 \rangle$, $V_5(k_2) = \langle \varepsilon_2 \rangle$. Let

$$E_1 := \mathrm{Spl}_{\mathbb{Q}} f(X, \varepsilon_1) \quad \text{and} \quad E_2 := \mathrm{Spl}_{\mathbb{Q}} f(X, \varepsilon_2).$$

Then, in virtue of $\delta_1 = \delta_2 = 0$, $E_1$ and $E_2$ are unramified cyclic quintic extensions of $M$. According to [9, p. 17, l. 9–23],

$$\mathrm{Gal}(E_1/\mathbb{Q}) \simeq F_{5,2} \quad \text{and} \quad \mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3}.$$

Let $L := E_1 \cdot E_2$ be the compositum. Then $E/M$ is also an unramified cyclic quintic extension, for any proper subextension $E$ of $L/M$ distinct from $E_1$ and $E_2$. Assume that $\mathrm{Gal}(E/\mathbb{Q}) \simeq F_{5,2}$. Since $L = E_1 \cdot E$, all proper subextensions $E'$ of $L/M$ have $\mathrm{Gal}(E'/\mathbb{Q}) \simeq F_{5,2}$, by [9, Lem. 2.5]. This is a contradiction to $\mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3}$. In the same manner, the assumption that $\mathrm{Gal}(E/\mathbb{Q}) \simeq F_{5,3}$ leads to a contradiction. Therefore $E/\mathbb{Q}$ must be a non-Galois extension. □

### 3.3 Infinite family of imaginary cyclic quartic fields *M* whose 5-rank is at least 2

As before, let $d \neq 1$ be a square-free integer prime to 5, and let $k_1 = \mathbb{Q}(\sqrt{d})$ and $k_2 = \mathbb{Q}(\sqrt{5d})$ be the associated quadratic fields. For $\gamma \in k = k_i,\ i \in \{1, 2\}$, Kishi [9, p. 6] has defined the polynomial

$$f(X, \gamma) = X^5 - 5\mathrm{N}_k(\gamma)X^3 + 5\mathrm{N}_k(\gamma)^2 X - \mathrm{N}_k(\gamma)^2\mathrm{Tr}_k(\gamma),$$

where $\mathrm{N}_k$, $\mathrm{Tr}_k$ are the norm map and the trace map of $k/\mathbb{Q}$. The minimal splitting field of $f(X, \gamma)$ is noted by $K_\gamma$. Furthermore, Imaoka and Kishi [7], have characterized all $F_{5,w}$-extensions with $w \in \{2, 3\}$ as $K_\gamma$ for a suitable elements $\gamma \in k_i$ with $i \in \{1, 2\}$. If $\gamma \in k_1$, then $\mathrm{Gal}\left(K_\gamma/\mathbb{Q}\right) \simeq F_{5,2}$, and if $\gamma \in k_2$, then $\mathrm{Gal}\left(K_\gamma/\mathbb{Q}\right) \simeq F_{5,3}$. Now, we consider the real quadratic fields $k_1 = \mathbb{Q}(\sqrt{d})$ and $k_2 = \mathbb{Q}(\sqrt{5d})$, $d = (\alpha + \beta)^2 - 4$, given by Kishi in [10, Ex. 3.5, p. 489] for $p = 5$, where the pair of integers $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$ such that $\alpha \geq 2,\ \beta \geq 2$ satisfies the simultaneous conditions

$$\begin{cases} \alpha^2 - 5^3\beta^2 = 4, \\ \alpha + \beta \equiv 0 \pmod{5^2}. \end{cases} \tag{3.5}$$

***Remark 3.2*** The Pellian equation $\alpha^2 - 5^3\beta^2 = 4$ has infinitely many solutions $(\alpha, \beta)$, which correspond to the powers $\eta^n = \frac{\alpha+\beta\sqrt{5^3}}{2}$ of the normpositive fundamental unit

$$\eta = \frac{123 + 11\sqrt{5^3}}{2} = \frac{123 + 11 \cdot 5\sqrt{5}}{2}$$

of the suborder with conductor $f = 5$ of $\mathbb{Q}(\sqrt{5})$. The solution $(\alpha, \beta)$ satisfies the additional constraint $\alpha + \beta \equiv 0 \bmod 5^2$ in (3.5) if and only if $n = 7 + 5^2 k$ with an integer $k \geq 0$.

**Proposition 3.2** *Let*

$$M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{(\alpha + \beta)^2 - 4}\right),$$

*where* $\alpha$, $\beta$ *satisfy the conditions* (3.5). *Then the* 5*-rank of the class group of* $M$ *is greater than or equal to* 2.

***Proof*** Let

$$\epsilon_1 = \frac{\alpha + \beta + \sqrt{d}}{2}, \quad \text{resp.} \quad \epsilon_2 = \frac{\alpha + 5^3\beta + 5\sqrt{5d}}{2},$$

be an element of

$$k_1 = \mathbb{Q}\left(\sqrt{(\alpha + \beta)^2 - 4}\right), \quad \text{resp.} \quad k_2 = \mathbb{Q}\left(\sqrt{5((\alpha + \beta)^2 - 4)}\right).$$

According to [10, Ex. 3.5, p. 489], $\epsilon_1$ and $\epsilon_2$ are units of $k_1$ and $k_2$, respectively. They satisfy the conditions

$$\begin{cases} \mathrm{N}_{\mathbb{Q}(\sqrt{d})}\left(\epsilon_1^2\right) = \mathrm{N}_{\mathbb{Q}(\sqrt{5d})}\left(\epsilon_2\right) = 1, \\ \mathrm{Tr}_{\mathbb{Q}(\sqrt{d})}\left(\epsilon_1^2\right) \equiv \mathrm{Tr}_{\mathbb{Q}(\sqrt{5d})}\left(\epsilon_2\right) \equiv \pm 2 \pmod{5^3}. \end{cases} \tag{3.6}$$

By applying [10, Th. 1.1, p. 482, Prop. 3.1, p. 487], we prove that $K_{\epsilon_1^2}$ and $K_{\epsilon_2}$ are two different absolute Galois $F_5$-extensions, unramified over $M$; it suffices to show that $\epsilon_1^2$, resp. $\epsilon_2$, cannot be the fifth power of an element of $k_1$, resp. $k_2$.

According to [18, Lem. 1, p. 16 ], we have the following general fact: Let $p$ be a prime number and $\xi$ be an element of $\mathbb{Q}(\sqrt{\delta})$ such that $\xi = \frac{u+v\sqrt{\delta}}{2}$. If $0 < |v| < \frac{\delta^{(p-1)/2}}{2^{p-1}}$, then $\xi \notin \mathbb{Q}(\sqrt{\delta})^p$.

Let us apply this result to $\epsilon_2$ and $\epsilon_1^2$. By the assumptions (3.5), $\alpha + \beta = 5^2 c$, for some $c \geq 1$. Hence $(\alpha + \beta)^2 = 5^4 c^2$ and $5(\alpha + \beta)^2 = 5^5 c^2$. Furthermore, $5^5 c^2 \geq 5^5 > 36$ and thus $5(\alpha + \beta)^2 - 20 > 16$, whence

$$5d > 16, \ (5d)^2 > 16^2 \ \text{and} \ \frac{(5d)^2}{16} > 16.$$

Finally $5 < 16 < \frac{(5d)^2}{2^4}$, and if we put $v := 5$ and $\delta := 5d$, then $v < \frac{\delta^2}{2^4}$, whence $\epsilon_2$ cannot be the fifth power of an element in $k_2$.

For $\epsilon_1^2$, we express the square in the form

$$\epsilon_1^2 = \frac{\frac{(\alpha+\beta)^2+d}{2} + (\alpha + \beta)\sqrt{d}}{2}.$$

Moreover we have,

$$\alpha + \beta < \frac{d^2}{16} \iff 16(\alpha + \beta) < (\alpha + \beta)^4 - 8(\alpha + \beta)^2 + 16.$$

Put $u := \alpha + \beta$. Then

$$\alpha + \beta < \frac{d^2}{16} \iff u^4 - 8u^2 - 16u + 16 > 0.$$

Since $\alpha \geq 2$ and $\beta \geq 2$, it follows that $u \geq 3$, whence $\phi(u) = u^4 - 8u^2 - 16u + 16$ is positive. Thus we get $\alpha + \beta < \frac{d^2}{2^4}$, and putting $v := \alpha + \beta$ and $\delta := d$ we conclude that $\epsilon_1^2$ cannot be a fifth power in $k_1$ either. □

**Corollary 3.2** *Let*

$$M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{(\alpha + \beta)^2 - 4}\right),$$

*where the integers* $\alpha$, $\beta$ *satisfy the conditions* (3.5). *Let* $\varphi$ *denote the generator of* $\mathrm{Gal}\left(\mathbb{Q}(\sqrt{5})/\mathbb{Q}\right)$. *Assume that the* 5*-class group* $\mathrm{Cl}_5(M)$ *of* $M$ *is of type* $(5, 5)$. *Then* $M_5^{(1)}/M$ *contains six unramified cyclic quintic extensions* $E_i/M$*, which give rise to absolute extensions of degree* 20 *over* $\mathbb{Q}$*, ordered the following way:*

- $E_1 = K_{\epsilon_1^2}$ *of Type (I) with* $\mathrm{Gal}(E_1/\mathbb{Q}) \simeq F_{5,2}$*, the splitting field of the polynomial* $f(X, \epsilon_1^2) = X^5 - 5X^3 + 5X - (d + 2)$*;*
- $E_2 = K_{\epsilon_2}$ *of Type (II) with* $\mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3}$*, the splitting field of the polynomial* $f(X, \epsilon_2) = X^5 - 5X^3 + 5X - (\alpha + 5^3\beta)$*;*
- *the other four extensions* $E_3$, $E_4 = E_3^{\varphi}$, $E_5$, $E_6 = E_5^{\varphi}$*, which are non-Galois of Type (III) over* $\mathbb{Q}$ *and form two conjugate pairs.*

***Proof*** The claims are a consequence of Proposition 3.2, the formulas of (3.6) and the fact that $\mathrm{Tr}_{\mathbb{Q}(\sqrt{d})}(\epsilon_1^2) = d + 2$ and $\mathrm{Tr}_{\mathbb{Q}(\sqrt{5d})}(\epsilon_2) = \alpha + 5^3\beta$. □

**Table 3** Some possible 5-class ranks $r_1 := \varrho_5(k_1)$, $r_2 := \varrho_5(k_2)$ and invariants $\delta_1, \delta_2$ for the associated quadratic fields $k_1, k_2$ which are 5-dual to a real cyclic quartic field $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$, $d < 0$,with 5-class rank $r := \varrho_5(M) = 2$

| Case | $r_1$ | $\delta_1$ | $r_2$ | $\delta_2$ |
|---|---|---|---|---|
| (a) | 2 | 0 | 0 | 0 |
| (b) | 0 | 0 | 2 | 0 |
| (c) | 1 | 0 | 1 | 0 |
| (d) | 2 | 1 | 1 | 0 |
| (e) | 1 | 0 | 2 | 1 |

***Remark 3.3*** For $d > 0$, if the fundamental units of the real quadratic fields $k_i$, $i = 1, 2$, are 5-primary, then the field $M$ has a non-Galois unramified cyclic quintic extension of Type (III) [9]. In this case, there are four pairwise conjugate extensions of Type (III), and among the remaining two Frobenius extensions one is of Type (I) and one is of Type (II) [9]. Note that in the case $d > 0$ the cyclic quartic field $M$ is imaginary. Also, if 5 divides the class number of $k_i$, $i = 1, 2$, there exists at most one 5-primary element of $k_i$, $i = 1, 2$, which gives rise to the Frobenius extensions of Type (I) and Type (II).

### 3.4 Real cyclic quartic fields $M$ with $d < 0$

As before, the two Frobenius groups $F_{5,w}$ of order 20 with primitive root $w \in \{2, 3\}$ modulo 5 will be denoted as in formula (3.4).

**Proposition 3.3** *Let $E_1, \ldots, E_6$ be the six unramified cyclic quintic extensions of the real cyclic quartic field $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$, $d < 0$, with 5-class group $\mathrm{Cl}_5(M) \simeq C_5 \times C_5$ or, more generally, of 5-class rank 2. The properties of these fields as absolute extensions $E_i/\mathbb{Q}$, in dependence on the five cases in Table 3, are given as follows:*

(1) *In case* (a)*, all six fields $E_1, \ldots, E_6$ are normal and share isomorphic automorphism groups $\mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,2}$, for $1 \le i \le 6$.*
(2) *In case* (b)*, all six fields $E_1, \ldots, E_6$ are normal and share isomorphic automorphism groups $\mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,3}$, for $1 \le i \le 6$.*
(3) *In all the other cases* (c), (d), (e)*, two extensions are normal with non-isomorphic automorphism groups, say $\mathrm{Gal}(E_1/\mathbb{Q}) \simeq F_{5,2}$ and $\mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3}$, but the other four extensions are non-Galois and form two conjugate pairs $E_3 \simeq E_4$ and $E_5 \simeq E_6$.*

***Proof*** Similar to the proof of Proposition 3.1. □

## 4 The second 5-class group $\mathrm{G}_5^{(2)}M$ of $M$

Based on the class number formula [11] for dihedral relative extensions $E$ of degree 10 over a base field $F$ with class number coprime to 5, we are now in a position to determine the isomorphism type of the Galois group $\mathrm{G}_5^{(2)}M = \mathrm{Gal}(M_5^{(2)}/M)$ of the second Hilbert 5-class field $M_5^{(2)}$ of a cyclic quartic field $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with 5-class group of type $(5, 5)$, because its unramified cyclic quintic extensions $E_i$, $1 \le i \le 6$, turn out to be relatively dihedral over the quadratic subfield $k_0^+ = \mathbb{Q}(\sqrt{5})$ of $M$, which has class number 1.

**Theorem 4.1** *The relation between the 5-class numbers $\mathrm{h}_5(E_i)$ of the six unramified cyclic quintic extensions $E_i$, $1 \le i \le 6$, of $M$ and the 5-class numbers $\mathrm{h}_5(L_i)$ of their non-Galois subfields $L_i$, which are of relative degree 5 over the field $k_0^+ = \mathbb{Q}(\sqrt{5})$, is given by*

$$\mathrm{h}_5(E_i) = \begin{cases} \mathrm{h}_5(L_i)^2 & \text{if } \#\ker(j_{E_i/M}) = 25, \quad (U_{k_0^+} : \mathrm{N}_{L_i/k_0^+}(U_{L_i})) = 1, \\ 5 \cdot \mathrm{h}_5(L_i)^2 & \text{if } \#\ker(j_{E_i/M}) = 25, \quad (U_{k_0^+} : \mathrm{N}_{L_i/k_0^+}(U_{L_i})) = 5, \\ 5 \cdot \mathrm{h}_5(L_i)^2 & \text{if } \#\ker(j_{E_i/M}) = 5, \quad (U_{k_0^+} : \mathrm{N}_{L_i/k_0^+}(U_{L_i})) = 1, \\ 25 \cdot \mathrm{h}_5(L_i)^2 & \text{if } \#\ker(j_{E_i/M}) = 5, \quad (U_{k_0^+} : \mathrm{N}_{L_i/k_0^+}(U_{L_i})) = 5, \end{cases} \tag{4.1}$$

*where $U_F$ denotes the unit group of a field $F$.*

***Proof*** According to Lemmermeyer [11, eq. (5.2), p. 685], we have the class number relation

$$\mathrm{h}_5(E_i) = \frac{(U_{k_0^+} : \mathrm{N}_{L_i/k_0^+}(U_{L_i}))}{\#\ker(j_{E_i/M})} \cdot \mathrm{h}_5(M) \cdot \mathrm{h}_5(L_i)^2,$$

where $\mathrm{h}_5(M) = 25$, due to our general assumption on $M$. Distinction between total principalization, $\#\ker(j_{E_i/M}) = 25$, and partial principalization, $\#\ker(j_{E_i/M}) = 5$, immediately yields the four claimed cases, in dependence on the unit norm indices $u_i := (U_{k_0^+} : \mathrm{N}_{L_i/k_0^+}(U_{L_i}))$. □

***Remark 4.1*** In order to prove Theorem 4.1 in a different manner, we can use the class number formula, due to Lemmermeyer [11, Th. 2.4, p. 681], and the following Lemma 4.1.

**Lemma 4.1** *Let $p$ be an odd prime and let $F$ be a number field with class number coprime to $p$. Let $k$ be a quadratic extension of $F$. Assume that $L$ is an unramified cyclic extension of $k$ of degree $p$. Then the extension $L/F$ is Galois, dihedral of degree $2p$, and we have the formula*

$$a := (U_L : U_K U_{K'} U_k) = \frac{\left(U_k : U_k^p\right)\left(U_F : \mathrm{N}_{K/F}(U_K)\right)}{\left(U_F : U_F^p\right)\left(U_k : \mathrm{N}_{L/k}(U_L)\right)}$$

*for the subfield unit index $a$, where $K \neq K'$ denote two conjugate non-Galois subfields of $L$.*

Since $p \ge 3$ is an odd prime and the existence of an unramified cyclic extension $L/k$ of degree $p$ excludes the irregular case $p = 3$, $F = \mathbb{Q}$, $k = \mathbb{Q}(\sqrt{-3})$ with $h_k = 1$, either both fields $k$ and $F$ contain the $p$th roots of unity or both not. Therefore,

$$\frac{\left(U_k : U_k^p\right)}{\left(U_F : U_F^p\right)} = p^{r(k)-r(F)}$$

with the torsion-free Dirichlet unit ranks $r(k)$ of $k$ and $r(F)$ of $F$. For an unramified extension $L/k$, the Theorem on the Herbrand quotient of $U_L$ is equivalent with $\#\ker(j_{L/k}) = p \cdot b$ with $b := \left(U_k : \mathrm{N}_{L/k}(U_L)\right)$. Using Lemma 4.1, which can be found in [11, p. 686], we can express the factor on the right hand side of the class number relation [11, Th. 2.4, p. 681],

$$\mathrm{h}_p(L) = \frac{a}{p^{1+r(k)-r(F)}} \cdot \mathrm{h}_p(k) \cdot \mathrm{h}_p(K)^2,$$

in the form

$$\frac{a}{p^{1+r(k)-r(F)}} = \frac{a \cdot \left(U_F : U_F^p\right)}{p \cdot \left(U_k : U_k^p\right)} = \frac{\left(U_F : \mathrm{N}_{K/F}(U_K)\right)}{\#\ker(j_{L/k})},$$

which we have used for $p = 5$, $F = k_0^+$, $k = M$, $L = E_i$, $K = L_i$ in the proof of Theorem 4.1.

### 4.1 Imaginary cyclic quartic fields *M* with *d* > 0

**Theorem 4.2** *The* 5*-class field tower of* $M$ *has length* $\ell_5 M = 1$ *if and only if the second* 5*-class group* $\mathrm{G}_5^{(2)} M$ *of* $M$ *is the abelian* 5*-group* $\langle 25, 2\rangle$ *of type* $(5, 5)$*. In this case,*

(1) *the* 5*-class groups* $\mathrm{Cl}_5(E_i)$ *are cyclic of order* 5*, for* $1 \le i \le 6$*,*
(2) *the* 5*-class groups* $\mathrm{Cl}_5(L_i)$ *are trivial, for* $1 \le i \le 6$*,*
(3) *the* 5*-principalization of* $M$ *is of type* a.1*,* $\varkappa(M) = (000000)$.

***Proof*** For $\mathrm{G}_5^{(2)} M \simeq \langle 25, 2\rangle$, we have the cyclic 5-class groups $\mathrm{Cl}_5(E_i) \simeq C_5$ and six total principalizations $\#\ker(j_{E_i/M}) = 25$. According to Theorem 4.1, we obtain $\mathrm{h}_5(E_i) = 5 = u_i \cdot \mathrm{h}_5(L_i)^2$, which enforces $\mathrm{h}_5(L_i) = 1$ and $u_i = 5$, for all $1 \le i \le 6$. □

***Example 4.1*** The values $d = 4357$ and $d = 4444$ give rise to fields $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with 5-class group of type $(5, 5)$ having a single-stage 5-class tower. Fields of this type are extremely rare, since they form a fraction of $\frac{2}{83}$ among the fields with $0 < d < 10000$. Therefore, only about 2% of the cases possess a single-stage tower.

**Proposition 4.1** *Let* $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$*, with* $d > 0$*, be an imaginary cyclic quartic field with* 5*-class group of type* $(5, 5)$*. Let* $E_i$*,* $1 \le i \le 6$*, be the six unramified cyclic quintic extensions of* $M$ *and* $L_i$ *their non-Galois subfields of relative degree* 5 *over the field* $k_0^+ = \mathbb{Q}(\sqrt{5})$*. Then the following holds true for each* $1 \le i \le 6$*:*

(1) *the subfield unit indices* $a_i := \left(U_{E_i} : U_{L_i} U_{L_i'} U_M\right)$ *are equal to* 1*,*
(2) *the unit norm indices* $u_i$ *satisfy the equivalence* $u_i = 1 \iff \#\ker(j_{E_i/M}) = 5$*,*
(3) *the relations between the* 5*-class numbers* $\mathrm{h}_5(E_i)$ *and* $\mathrm{h}_5(L_i)$ *are given by*

$$\mathrm{h}_5(E_i) = 5 \cdot \mathrm{h}_5(L_i)^2.$$

***Proof*** According to Lemma 4.1, we can deduce that

$$a_i b_i = \frac{(U_M : U_M^5)\left(U_{k_0^+} : \mathrm{N}_{L_i/k_0^+}(U_{L_i})\right)}{\left(U_{k_0^+} : U_{k_0^+}^5\right)},$$

where $b_i$ denotes the unit norm index $\left(U_M : \mathrm{N}_{E_i/M}(U_{E_i})\right)$. Since $d > 0$, the field $M$ is imaginary and it is a CM-field with maximal real subfield $M^+ = k_0^+$. Hence, the torsion-free Dirichlet unit rank of $M$ is $r(M) = 1$, and $U_M = \langle -1, \epsilon_5\rangle$, where $\epsilon_5$ denotes the fundamental unit of the quadratic field $k_0^+ = \mathbb{Q}(\sqrt{5})$. This implies that

$$\left(U_M : U_M^5\right) = \left(U_{k_0^+} : U_{k_0^+}^5\right) \quad \text{and} \quad a_i b_i = u_i.$$

(1) To prove the first assertion, it suffices to show the following equivalence:

$$u_i = 1 \ \text{ if and only if } \quad b_i = 1.$$

So it suffices to show that the fundamental unit $\epsilon_5$ of $k_0^+$, which is also the fundamental unit of $M$, is the norm of a unit of $E_i$ if and only if it is the norm of a unit of $L_i$. If $u_i = 1$,

for $i \in \{1, \dots, 6\}$, then $\epsilon_5$ is the norm of a unit of $L_i$ (a non-Galois subfield of $E_i$), hence it is also the norm of the same unit in $E_i$, and $b_i = 1$.
Now suppose that $b_i = 1$, for $1 \leq i \leq 6$. Then there exists a unit $\xi \in U_{E_i}$ such that $\epsilon_5 = \mathrm{N}_{E_i/M}(\xi)$, and we obtain the following chain of implications:

$$\begin{aligned}\mathrm{N}_{M/\mathbb{Q}(\sqrt{5})}(\epsilon_5) = \mathrm{N}_{M/\mathbb{Q}(\sqrt{5})}&\left(\mathrm{N}_{E_i/M}(\xi)\right)\\ &\Rightarrow \epsilon_5^2 = \mathrm{N}_{M/\mathbb{Q}(\sqrt{5})}\left(\mathrm{N}_{E_i/M}(\xi)\right) = \mathrm{N}_{L_i/\mathbb{Q}(\sqrt{5})}\left(\mathrm{N}_{E_i/L_i}(\xi)\right)\\ &\Rightarrow \epsilon_5^6 = \mathrm{N}_{L_i/\mathbb{Q}(\sqrt{5})}\left(\mathrm{N}_{E_i/L_i}(\xi^3)\right)\\ &\Rightarrow \epsilon_5 \cdot \mathrm{N}_{L_i/\mathbb{Q}(\sqrt{5})}(\epsilon_5) = \mathrm{N}_{L_i/\mathbb{Q}(\sqrt{5})}\left(\mathrm{N}_{E_i/L_i}(\xi^3)\right),\end{aligned}$$

whence

$$\epsilon_5 = \mathrm{N}_{L_i/\mathbb{Q}(\sqrt{5})}\left(\epsilon_5^{-1} \cdot \mathrm{N}_{E_i/L_i}(\xi^3)\right).$$

Since the element $\epsilon_5^{-1} \cdot \mathrm{N}_{E_i/L_i}(\xi^3)$ is a unit of $L_i$, we obtain the index $u_i = 1$. On the other hand, the possible values of $b_i$ and $u_i$ are $\{1, 5\}$, and we can deduce that $u_i = b_i$. Finally, it follows from the equation $a_i b_i = u_i$ that $a_i = 1$.

(2) The result follows immediately from the fact that $\#\ker(j_{E_i/M}) = 5 \cdot b_i$.
(3) According to Theorem 4.1, we have two possible cases,

$$u_i = 1 \text{ and } \#\ker j_{E_i/M} = 5,$$

and

$$u_i = 5 \text{ and } \#\ker j_{E_i/M} = 25.$$

In both cases, the class number formula is given by $\mathrm{h}_5(E_i) = 5 \cdot \mathrm{h}_5(L_i)^2$.

□

**Theorem 4.3** *Let* $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ *with* $d > 0$ *be an imaginary cyclic quartic field with* 5*-class group* $\mathrm{Cl}_5(M) \simeq C_5 \times C_5$*. If the second* 5*-class group* $G := \mathrm{G}_5^{(2)}M$ *of* $M$ *is non-abelian, then the coclass* $\mathrm{cc}(G)$ *of* $G$ *is greater than or equal to* 2, $\mathrm{cc}(G) \geq 2$.

***Proof*** Assume that $G$ is non-abelian of coclass $cc(G) = 1$. Then the possible capitulation types of $M$ in the six intermediary cyclic quintic extensions of $M_5^{(2)}/M$, noted by

$$E_1,\ E_2,\ E_3,\ E_4 = E_3^{\varphi},\ E_5,\ E_6 = E_5^{\varphi},$$

are given by $\varkappa(G) = (111111)$ or $\varkappa(G) = (\ell 00000)$, $\ell \in \{0, 1, 2\}$.

First we consider the type $\varkappa(G) = (111111)$. In this case, the group $G$ is the extra special 5-group of order $5^3$ and exponent $5^2$, whose maximal normal subgroups are of order $5^2$. This implies that the 5-class number of each $E_i$ is equal to $5^2$. Using Proposition 4.1, however, we conclude that the valuation $v_5(\mathrm{h}_5(E_i))$ of the 5-class number of $E_i$ must be odd, which is a contradiction. Thus the type $\varkappa(G) = (111111)$ cannot occur.

For the three other types, we have total capitulation in the five extensions

$$E_2,\ E_3,\ E_4 = E_3^{\varphi},\ E_5,\ E_6 = E_5^{\varphi},$$

so the value of the index $b_i$, $2 \leq i \leq 6$, is $b_i = 5$, whence $u_i = 5$. On the other hand, for $2 \leq i \leq 6$ we again have $\mathrm{h}_5(E_i) = 5^2$, which is a contradiction, since by Proposition 4.1, the valuation $v_5(\mathrm{h}_5(E_i))$ must be odd. □

**Proposition 4.2** (Number of fields) *In the range* $0 < d < 10000$ *of fundamental discriminants* $d$ *of real quadratic fields* $k_1 = \mathbb{Q}(\sqrt{d})$ *with* $\gcd(5, d) = 1$, *there exist precisely* **83** *cases such that the* 5*-dual field* $M = \mathbb{Q}((\zeta_5 - \zeta_5^{-1})\sqrt{d})$ *of* $k_1$ *has a* 5*-class group* $\mathrm{Cl}_5(M)$ *of type* $(5, 5)$.

***Proof*** See Tables 4 and 5. □

**Theorem 4.4** (Two-stage towers of 5-class fields with Schur $\sigma$-groups)

(1) *If the* 5*-dual field* $M$ *of* $k_1$ *has* 5*-principalization type* $\varkappa(M) = (125643)$ *with two fixed points and a* 4*-cycle, then the abelian type invariants of* $E_1, \dots, E_6$ *are* $\tau(M) = [(1^3)^2, (21)^4]$, *and the* 5*-class tower group is the Schur* $\sigma$*-group* $\mathrm{G}_5^{(\infty)} M = \mathrm{G}_5^{(2)} M \simeq \langle 5^5, \mathbf{11}\rangle$.

(2) *If the* 5*-dual field* $M$ *of* $k_1$ *has* 5*-principalization type* $\varkappa(M) = (123456)$, *the identity permutation, then the abelian type invariants of* $E_1, \dots, E_6$ *are* $\tau(M) = [(1^3)^6]$, *and the* 5*-class tower group is the Schur* $\sigma$*-group* $\mathrm{G}_5^{(\infty)} M = \mathrm{G}_5^{(2)} M \simeq \langle 5^5, \mathbf{14}\rangle$.

***Proof*** In each case, the length of the 5-class tower of $M$ is given by $\ell_5 M = 2$, since $G := \mathrm{G}_5^{(2)} M$ is a Schur $\sigma$-group with balanced presentation, i.e., relation rank $d_2(G) = d_1(G) = \varrho_5(M) = 2$. □

Examples for part (1) are the **23** (about **28**%) real quadratic fields $k_1$ starting with the following discriminants:

$$d \in \{457,\ 501,\ 1996,\ 2573,\ 3253,\ 4189,\ 4957,\ 5129,\ 5233,\ 5308,\ 5361,\ \dots\}.$$

Examples for part (2) are the **11** (about **13**%) real quadratic fields $k_1$ with the following discriminants:

$$d \in \{581,\ 753,\ 2296,\ 2829,\ 4553,\ 5116,\ 5736,\ 6761,\ 7489,\ 9013,\ 9829\},$$

verifying a conjecture by O. Taussky in [22], and announced in [13, Sect. 3.5.2, p. 448], except 2829.

***Remark 4.2*** The pairs of conjugate non-Galois extensions $E_3 \simeq E_4$ and $E_5 \simeq E_6$ of $M$ are not adjacent in the factor (3546) of the cycle pattern (1)(2)(3546) of the 4-cycle $\varkappa(M) = (125643)$, and the Frobenius extensions $E_1, E_2$ correspond to the fixed points (1), (2). The identity $\varkappa(M) = (123456)$, which does not have two distinguished fixed points a priori, is endowed with a random arithmetical bipolarization by the two Frobenius extensions $E_1, E_2$.

**Theorem 4.5** (Two-stage towers of 5-class fields with unusual capable weak $\sigma$-groups)

(1) *If the* 5*-dual field* $M$ *of* $k_1$ *has* 5*-principalization type* $\varkappa(M) = (022222)$, *nearly constant with a single total capitulation and a single fixed point, then the abelian type invariants of* $E_1, \dots, E_6$ *are* $\tau(M) = [(1^3)^2, (21)^4]$, *and the* 5*-tower group is* $\mathrm{G}_5^{(\infty)} M = \mathrm{G}_5^{(2)} M \simeq \langle 5^5, \mathbf{4}\rangle$.

(2) *If the* 5*-dual field* $M$ *of* $k_1$ *has* 5*-principalization type* $\varkappa(M) = (124365)$ *with two fixed points and two disjoint* 2*-cycles, then the abelian type invariants of* $E_1, \dots, E_6$ *are* $\tau(M) = [(1^3)^2, (21)^4]$, *and the* 5*-class tower group is* $\mathrm{G}_5^{(\infty)} M = \mathrm{G}_5^{(2)} M \simeq \langle 5^5, \mathbf{7}\rangle$.

***Proof*** In each case, the length of the 5-class tower of $M$ is given by $\ell_5 M = 2$, since $G := \mathrm{G}_5^{(2)} M$ is a metabelian $\sigma$-group with trivial cover [15, Def. 5.1, p. 30], according to Heider and Schmithals [6, p. 20]. The presentation of $G$ is not balanced, since the relation rank

$d_2(G) = 3$ is too big. However, the Shafarevich Theorem [20], in its corrected version [15, Th. 5.1, p. 28], ensures that $d_2(G) \le d_1(G) + r = 3$, (it just reaches the admissible upper bound), since the generator rank of $G$ and the torsion-free Dirichlet unit rank of $M$ with signature $(0, 2)$ are given by $d_1(G) = \varrho_5(M) = 2$ and $r = 0 + 2 - 1 = 1$. The 5-tower groups $\langle 5^5, \mathbf{4}\rangle$ and $\langle 5^5, \mathbf{7}\rangle$ are unusual, because they are not strong $\sigma$-groups and thus are forbidden for (imaginary and real) quadratic base fields [19]. □

Examples for part (1) are the **22** (about **27**%) real quadratic fields $k_1$ starting with the following discriminants:

$$d \in \{257,\ 764,\ 1708,\ 1853,\ 2008,\ 2189,\ 3129,\ 4504,\ 4861,\ 5241,\ 5269,\ \ldots\}.$$

Examples for part (2) are the **16** (about **19**%) real quadratic fields $k_1$ starting with the following discriminants:

$$d \in \{508,\ 509,\ 629,\ 881,\ 1113,\ 1192,\ 1704,\ 1829,\ 3121,\ 4461,\ 7032,\ \ldots\}.$$

***Remark 4.3*** The pairs of conjugate non-Galois extensions $E_3 \simeq E_4$ and $E_5 \simeq E_6$ of $M$ correspond to the factors (34) and (56) of the cycle pattern (1)(2)(34)(56) of the two disjoint 2-cycles $\varkappa(M) = (124365)$, and the Frobenius extensions $E_1, E_2$ correspond to the fixed points (1), (2). For the nearly constant type $\varkappa(M) = (022222)$, the first (resp. second) Frobenius extension $E_1$ (resp. $E_2$) corresponds to the single total capitulation (resp. the single fixed point).

Figure 3 visualizes the situation of a two-stage 5-class tower in the Theorems 4.4, 4.5, 4.9.

**Theorem 4.6** (Single-stage towers of 5-class fields with abelian group) *For the* **2** *(about* **2**%*) real quadratic fields $k_1$ with discriminants $d \in \{4357,\ 4444\}$, the 5-dual field $M$ of $k_1$ has 5-principalization type $\varkappa(M) = (000000)$, a constant with six total capitulations; the abelian type invariants of $E_1, \ldots, E_6$ are $\tau(M) = [(1)^6]$, and thus the 5-class tower is abelian with group $\mathrm{G}_5^{(\infty)} M = \mathrm{G}_5^{(1)} M \simeq \langle 5^2, \mathbf{2}\rangle$ and length $\ell_5 M = 1$.*

***Proof*** Here, the 5-class tower is abelian with length $\ell_5 M = 1$, according to Theorem 4.2. □

***Remark 4.4*** Outside the range $0 < d < 10^4$ of our systematic investigations, we have discovered three occurrences of case (g) in Table 2. For the real quadratic fields $k_1$ with discriminants $d \in \{244641,\ 1277996,\ 1915448\}$ the 5-dual field $M$ of $k_1$ has 5-principalization type $\varkappa(M) = (000000)$, a constant with six total capitulations, abelian type invariants $\tau(M) = [(1)^6]$, and abelian 5-class tower with group $\mathrm{G}_5^{(\infty)} M = \mathrm{G}_5^{(1)} M \simeq \langle 5^2, \mathbf{2}\rangle$ and length $\ell_5 M = 1$. The invariants are given by $(r_1, r_2, \delta_1, \delta_2) = (2, 0, 1, 1)$.

**Theorem 4.7** (Frobenius and non-Galois extensions) *The properties of the absolute extensions $E_i/\mathbb{Q}$ and the values of the invariants in the Quintic Reflection Theorem, Table 2, and Proposition 3.1, for the* **83** *cases in Proposition 4.2 are the following ones:*

(i) *For the* **2** *cases with $\ell_5 M = 1$ in Theorem 4.6, we have*

$$(r_1, r_2, \delta_1, \delta_2) = (1, 0, 0, 1) \ \text{and}\ \mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,2} \text{ for } 1 \le i \le 6 \ \ (\text{Case (a)})$$

(ii) *For the other* **81** *cases, including the* **34** *cases of $\ell_5 M = 2$ in Theorem 4.4 and the* **38** *cases of $\ell_5 M = 2$ in Theorem 4.5, we have pairwise conjugate non-Galois extensions*

$$E_3 \simeq E_4,\ E_5 \simeq E_6 \ \text{with}\ \mathrm{Gal}(E_1/\mathbb{Q}) \simeq F_{5,2},\ \mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3}$$

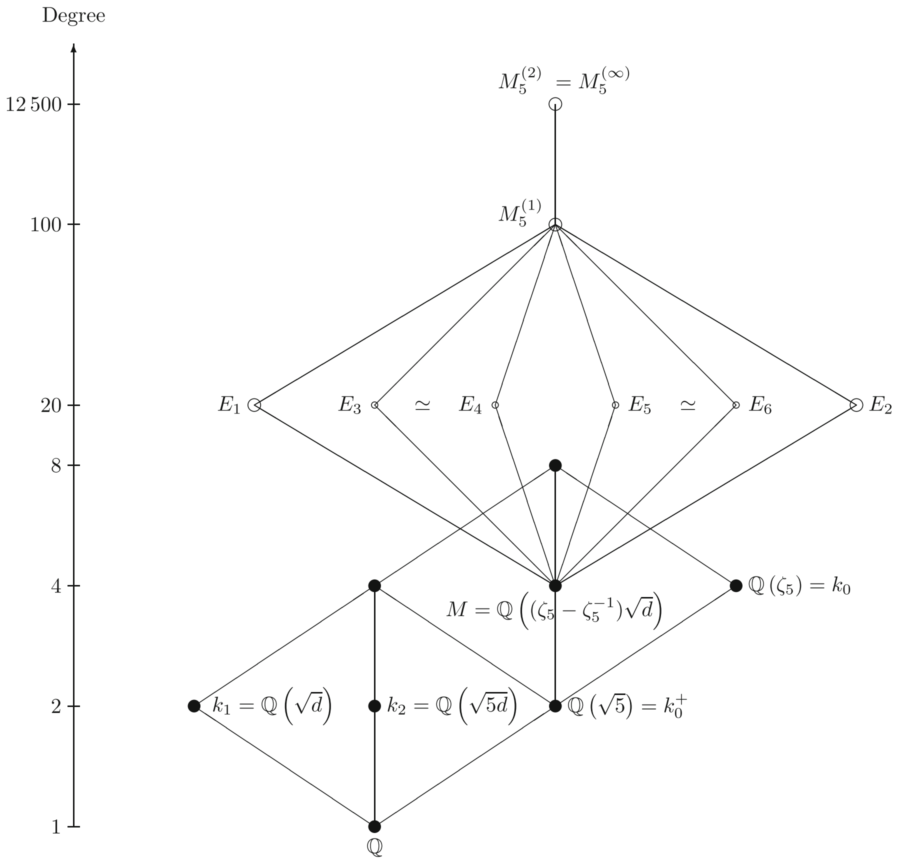


**Fig. 3** The 5-class tower $M_5^{(\infty)}$ of $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ when $\#\mathrm{G}_5^{(2)} M = 5^5$

*and*

$$\begin{cases} (r_1, r_2, \delta_1, \delta_2) = (1, 0, 1, 0), \ \textit{for } d \in \{1996,\ 3121,\ 3129,\ 3253,\ 5241,\ 5269, \\ \qquad\qquad\qquad\qquad 5308, 6113,\ 8309,\ 8689,\ 9829\} \ (\textit{Case } (\mathrm{e})), \\ (r_1, r_2, \delta_1, \delta_2) = (0, 1, 0, 1), \ \textit{for } d \in \{5116,\ 8972,\ 9013\} \ (\textit{Case } (\mathrm{f})), \\ (r_1, r_2, \delta_1, \delta_2) = (1, 1, 1, 1), \ \textit{for } d \in \{4504,\ 6949,\ 7221,\ 7229,\ 9669\} \ (\textit{Case } (\mathrm{c})), \\ (r_1, r_2, \delta_1, \delta_2) = (0, 0, 0, 0), \ \textit{otherwise } (\textit{Case } (\mathrm{d})). \end{cases}$$

***Proof*** See Tables 4 and 5. □

## 4.2 Real cyclic quartic fields *M* with *d* < 0

**Proposition 4.3** *Let $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $d < 0$ be a real cyclic quartic field with 5-class group of type $(5, 5)$. Denote by $E_i$, $1 \le i \le 6$, the six unramified cyclic quintic extensions of $M$ and by $L_i$ their non-Galois subfields, which are of relative degree 5 over the field $k_0^+ = \mathbb{Q}(\sqrt{5})$.*

(1) *If* $\#ker(j_{E_i/M}) = 5$*, then the unit norm index is*

$$u_i = (U_{k_0^+} : \mathrm{N}_{L_i/k_0^+}(U_{L_i})) = 1,$$

*and in this case the subfield unit index* $a_i = \left(U_{E_i} : U_{L_i}U_{L_i'}U_M\right)$ *is equal to* 25.

(2) *The relation between the* 5*-class numbers* $\mathrm{h}_5(E_i)$ *and* $\mathrm{h}_5(L_i)$ *is given by*

$$\mathrm{h}_5(E_i) = \begin{cases} 5 \cdot \mathrm{h}_5(L_i)^2 & \text{if } b_i = u_i, \\ \mathrm{h}_5(L_i)^2 & \text{if } b_i \neq u_i, \text{ where } b_i = \left(U_M : \mathrm{N}_{E_i/M}(U_{E_i})\right). \end{cases}$$

***Proof*** Since $d < 0$, the field $M$ is totally real, and the Dirichlet rank of its torsion-free unit group is given by $r(M) = 3$.

(1) Denote by $U_{M/\mathbb{Q}(\sqrt{5})}$ the group of relative units

$$\{\epsilon \in U_M \mid \mathrm{N}_{M/\mathbb{Q}(\sqrt{5})}(\epsilon) = 1\}.$$

For a cyclic quartic field $K/\mathbb{Q}$ with real quadratic subfield $k$, Hasse showed that the group $U_kU_{K/k}$ has index at most 2 in the full group of units $U_K$. In our case, $U_{\mathbb{Q}(\sqrt{5})}U_{M/\mathbb{Q}(\sqrt{5})}$ has index at most 2 in $U_M$, where

$$U_M = \langle -1, \epsilon_5, \eta, \eta^{\mu\tau} \rangle,$$

with $\eta$ satisfying $\eta^{1+\mu\tau} = \pm 1$. If $\#\ker(j_{E_i/M}) = 5$, which means that the unit norm index $b_i = (U_M : \mathrm{N}_{E_i/M}(U_{E_i}))$ is equal to 1, then all units of $M$ are the norms of a unit of $E_i$, in particular $\epsilon_5$. In the same manner as in the proof of claim 1 of Proposition 4.1, we deduce that $\epsilon_5$ is also the norm of a unit of $L_i$, whence $u_i = 1$. On the other hand, by applying Lemma 4.1, we deduce that $a_i \cdot b_i = 25 \cdot u_i$, and consequently $a_i = 25$.

(2) According to Theorem 4.1 or the Lemmermeyer class number formula [11, Th. 2.4, p. 681], we conclude that $\mathrm{h}_5(E_i) = 5 \cdot \mathrm{h}_5(L_i)^2$ if $b_i = 1$ or ($b_i = 5$ and $u_i = 5$). But if $b_i = 5$ and $u_i = 1$, we have $\mathrm{h}_5(E_i) = \mathrm{h}_5(L_i)^2$, which completes the proof.

□

***Remark 4.5*** For totally real or imaginary cyclic quartic fields $M$, the last case of Theorem 4.1 given by $\mathrm{h}_5(E_i) = 25 \cdot \mathrm{h}_5(L_i)^2$ is impossible for any $i \in \{1, 2, 3, 4, 5, 6\}$.

**Proposition 4.4** *Let* $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ *with* $d < 0$ *be a real cyclic quartic field with* 5*-class group of type* (5, 5)*. Let* $E_i$*,* $1 \leq i \leq 6$*, be the six unramified cyclic quintic extensions of* $M$*. Denote by* $G := \mathrm{G}_5^{(2)}M$ *the second* 5*-class group of* $M$ *and assume that the order of* $G$ *is equal to* $|G| = 5^3$*. Then the transfer kernel type of* $G$ *is* $\varkappa(G) = (000000)$ *(capitulation type of* $M$ *in the six unramified extensions* $E_i$*) and the transfer target type of* $G$ *is* $\tau(G) = \left[(1^2)^6\right]$.

***Proof*** In this case, the group $G$ is extra special of maximal class. Thus, the possible types of capitulation are (111111) and (000000). First, we know that the type (111111) is not possible, because in this case $\mathrm{h}_5(E_i) = 5^2$ and $b_i = 1$, which contradicts claim (2) of Proposition 4.3. Thus, the transfer kernel type of $G$ is $\varkappa(G) = (000000)$.

On the other hand, for all $1 \leq i \leq 6$, the unit norm index is $b_i = 5$, and $u_i$ must be equal to 1. Otherwise, the Lemmermeyer class number formula [11, Th. 4.1, p. 456] implies $|G| \geq 5^4$. Thus, for all $1 \leq i \leq 6$, we have $\mathrm{h}_5(E_i) = \mathrm{h}_5(L_i)^2$ and $\mathrm{h}_5(L_i) = 5$. Since the six extensions $E_i$ are of type $A$ in the sense of Taussky and $\mathrm{h}_5(E_i) = 5^2$, we deduce that $\mathrm{Cl}_5(E_i)$ is of type (5, 5). Thus $\tau(G) = \left[(1^2)^6\right]$. □

***Remark 4.6*** Assume that the group $G$ is not abelian and $\varkappa(G) = (000000)$. Then the prime 5 must divide the class number of the fields $L_i$. Because, in this case $b_i = 5$ and $u_i = 1$ or 5. The case $u_i = 1$ is obvious. Now suppose that $u_i = 5$. If 5 does not divide $\mathrm{h}(L_i)$, then $\mathrm{h}_5(E_i) = 5$, Hence $E_i$ is an unramified extension of $M$ and $\mathrm{h}_5(E_i) = \frac{\mathrm{h}_5(M)}{5}$. Then $M_5^{(2)} = M_5^{(1)}$ and the group $G$ is abelian, which is a contradiction.

**Proposition 4.5** (Number of fields) *In the range* $-200000 < d < 0$ *of fundamental discriminants* $d$ *of imaginary quadratic fields* $k_1 = \mathbb{Q}(\sqrt{d})$ *with* $\gcd(5, d) = 1$, *there exist precisely* **93** *cases such that the* 5*-dual field* $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ *of* $k_1$ *has a* 5*-class group* $\mathrm{Cl}_5(M)$ *of type* $(5, 5)$.

***Proof*** See Tables 6, 7 and 8. □

**Theorem 4.8** (Two-stage towers of 5-class fields with groups of low order)

(1) *If the* 5*-dual field* $M$ *of* $k_1$ *has* 5*-principalization type* a.1, $\varkappa(M) = (000000)$, *a constant with six total capitulations, and the abelian type invariants of* $E_1, \ldots, E_6$ *are* $\tau(M) = [(1^2)^6]$, *then the* 5*-tower group is the extra special group* $\mathrm{G}_5^{(\infty)}M = \mathrm{G}_5^{(2)}M \simeq \langle 5^3, \mathbf{3}\rangle$.
(2) *If the* 5*-dual field* $M$ *of* $k_1$ *has* 5*-principalization type* a.2, $\varkappa(M) = (100000)$ *with a fixed point and five total capitulations, and the abelian type invariants of* $E_1, \ldots, E_6$ *are* $\tau(M) = [1^3, (1^2)^5]$, *then the* 5*-tower group is the Schur*$+1$ $\sigma$*-group* $\mathrm{G}_5^{(\infty)}M = \mathrm{G}_5^{(2)}M \simeq \langle 5^4, \mathbf{8}\rangle$.
(3) *If the* 5*-dual field* $M$ *of* $k_1$ *has* 5*-principalization type* a.1, $\varkappa(M) = (000000)$, *a constant with six total capitulations, and the abelian type invariants of* $E_1, \ldots, E_6$ *are* $\tau(M) = [1^3, (1^2)^5]$, *then the* 5*-tower group is the mainline group* $\mathrm{G}_5^{(\infty)}M = \mathrm{G}_5^{(2)}M \simeq \langle 5^4, \mathbf{7}\rangle$.

***Proof*** In each case, the length of the 5-class tower of $M$ is given by $\ell_5 M = 2$, according to Blackburn [3], since $G := \mathrm{G}_5^{(2)}M$ is a $\sigma$-group with at most two-generated commutator subgroup $G' \in \{1, 1^2\}$. The presentation of $G$ is not balanced, since the relation rank $d_2(G) \in \{3, 4\}$ is too big. However, the Shafarevich Theorem [20] in its corrected version [15, Th. 5.1, p. 28] ensures that $d_2(G) \le d_1(G) + r = 5$ does not exceed the admissible upper bound, since the generator rank of $G$ and the torsion-free Dirichlet unit rank of $M$ with signature $(4, 0)$ are given by $d_1(G) = \varrho_5(M) = 2$ and $r = 4 + 0 - 1 = 3$. □

Examples for Case (1) are **56** (about **60**%) imaginary quadratic fields $k_1$ starting with the discriminants

$$d \in \{-12883,\ -13147,\ -14339,\ -23336,\ -23732,\ -26743, -28696,\ -35067,\ -35839,\ -38984,\ -47172,\ \ldots\}.$$

Examples for Case (2) are **23** (about **25**%) imaginary quadratic fields $k_1$ starting with the discriminants

$$d \in \{-27528,\ -27939,\ -39947,\ -40823,\ -54347,\ -75892,\ -91127,\ -99428,\ -101784,\ -105431,\ -114679,\ \ldots\}.$$

Examples for Case (3) are **8** (about **9**%) imaginary quadratic fields $k_1$ with the following discriminants

$$d \in \{-15419,\ -16724,\ -31103,\ -42899,\ -67128,\ -70763,\ -105784,\ -194487\}.$$

**Theorem 4.9** (Two-stage tower of 5-class fields with Schur $\sigma$-group) *If the 5-dual field $M$ of $k_1$ has 5-principalization type $\varkappa(M) = (124563)$, a 4-cycle and two fixed points, then the abelian type invariants of $E_1, \ldots, E_6$ are $\tau(M) = [(1^3)^2, (21)^4]$ and the 5-class tower group is $\mathrm{G}_5^{(\infty)} M = \mathrm{G}_5^{(2)} M \simeq \langle 5^5, \mathbf{11}\rangle$. In this case, the length of the 5-class tower of $M$ is given by $\ell_5 M = 2$, and $G := \mathrm{G}_5^{(2)} M$ is a Schur $\sigma$-group with balanced presentation, that is, relation rank $d_2(G) = d_1(G) = \varrho_5(M) = 2$.*

***Proof*** Similar to the proof of Theorem 4.4. □

The *unique* example is the imaginary quadratic field $k_1$ with discriminant $d = -114303$.

**Theorem 4.10** (Single-stage towers of 5-class fields with abelian group) *For the **5** (about **5**%) imaginary quadratic fields $k_1$ with discriminants*

$$d \in \{-58424,\ -115912,\ -148507,\ -151879,\ -154408\},$$

*the 5-dual field $M$ of $k_1$ has 5-principalization type $\varkappa(M) = (000000)$, a constant with six total capitulations, the abelian type invariants of $E_1, \ldots, E_6$ are $\tau(M) = [(1)^6]$, and the 5-class tower is abelian with group $\mathrm{G}_5^{(\infty)} M = \mathrm{G}_5^{(1)} M \simeq \langle 5^2, \mathbf{2}\rangle$ and length $\ell_5 M = 1$.*

***Proof*** Similar to the proof of Theorem 4.6. □

**Theorem 4.11** (Frobenius and non-Galois extensions) *The properties of the absolute extensions $E_i/\mathbb{Q}$ and the values of the invariants in the Quintic Reflection Theorem, Table 3, and Proposition 3.3, for the **93** cases in Proposition 4.5 are the following ones:*

(1) *For the **5** cases with $\ell_5 M = 1$ in Theorem 4.10, we have*

$$(r_1, r_2, \delta_1, \delta_2) = (2, 0, 0, 0),\ \textit{and}\ \mathrm{Gal}(E_i/\mathbb{Q}) \simeq F_{5,2}\ \textit{for}\ 1 \le i \le 6\ (\textit{Case}\ \text{(a)}).$$

(2) *For the other **88** cases, including the **87** cases of $\ell_5 M = 2$ in Theorem 4.8, and the **unique** case of $\ell_5 M = 2$ in Theorem 4.9, we have pairwise conjugate non-Galois extensions*

$$E_3 \simeq E_4,\ E_5 \simeq E_6,\ \mathrm{Gal}(E_1/\mathbb{Q}) \simeq F_{5,2},\ \ \mathrm{Gal}(E_2/\mathbb{Q}) \simeq F_{5,3},$$

*and*

$$\begin{cases} (r_1, r_2, \delta_1, \delta_2) = (2, 1, 1, 0),\ \textit{for}\ d \in \{-39947, -64103, -67128, -104503, -119191\} (\textit{Case}\ \text{(d)}), \\ (r_1, r_2, \delta_1, \delta_2) = (1, 2, 0, 1),\ \textit{for}\ d \in \{-110479,\ -199735\}\ (\textit{Case}\ \text{(e)}), \\ (r_1, r_2, \delta_1, \delta_2) = (1, 1, 0, 0),\ \textit{otherwise}\ (\textit{Case}\ \text{(c)}). \end{cases}$$

***Proof*** See Tables 6, 7 and 8. □

Figure 4 visualizes the relevant part of the descendant tree of finite 5-groups, beginning at the abelian root $C_5 \times C_5 = \langle 5^2, 2\rangle$, on which the second 5-class groups $\mathrm{G}_5^{(2)} M$ of the fields $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ are located as vertices. The figure is a modification of the diagram in [13, Fig. 3.8, p. 448]. The minimal positive, resp. maximal negative, discriminants $d$ are indicated by underlined boldface integers adjacent to the oval surrounding the vertex realized by $\mathrm{G}_5^{(2)} M$. The identifiers are due to the packages [2,4] which are implemented in [12]. (For trees, see [14].)

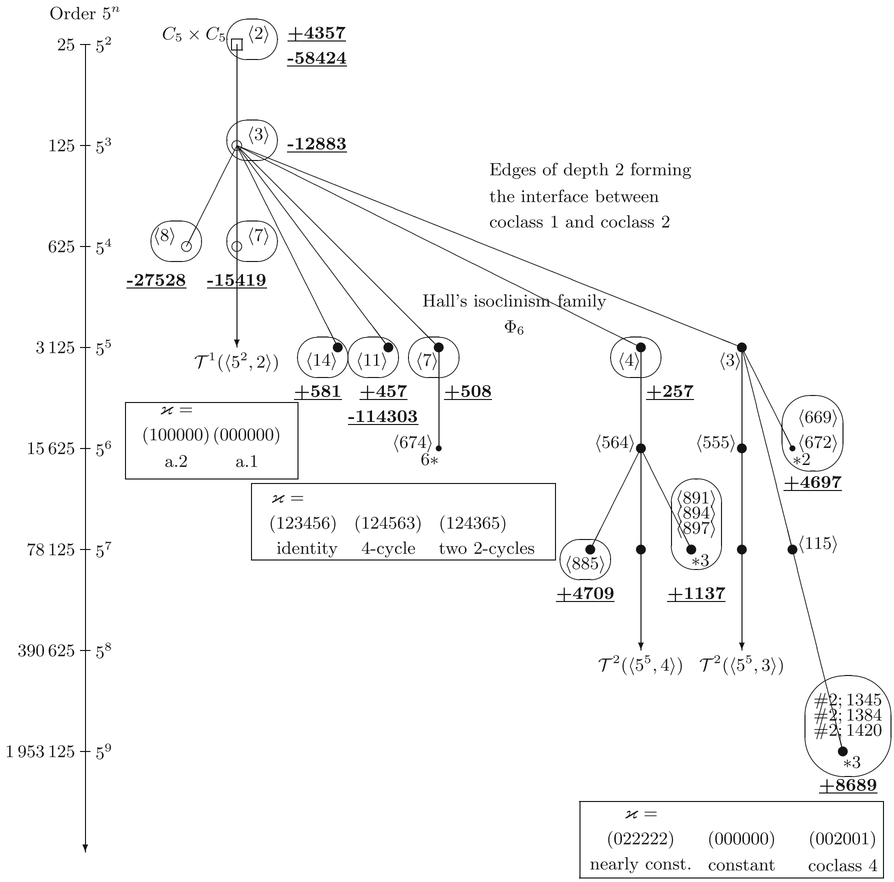


**Fig. 4** Tree position of second 5-class groups $\mathrm{G}_5^{(2)}M$ of the fields $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$

## 5 Tables of second 5-class groups $\mathrm{G}_5^{(2)}M$

### 5.1 Imaginary cyclic quartic fields $M$ with $d > 0$

Table 4, resp. Table 5, shows the factorized fundamental discriminant $d$ of the dual quadratic field $k_1$, the 5-principalization type $\varkappa = \varkappa(M)$, the second 5-class group $\mathrm{G}_5^{(2)}M$, the length $\ell_5 M$ of the 5-class tower, the 5-class ranks $r_1 := \varrho_5(k_1)$, $r_2 := \varrho_5(k_2)$, the invariants $\delta_1$, $\delta_2$, and the case in Proposition 3.1 for the 37, resp. 46, cyclic quartic fields $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $0 < d < 5000$, resp. $5000 < d < 10000$.

For the fields with constant 5-principalization type, consisting of partial kernels, we have a *polarization* of the target type whose abelian invariants can be either homogeneous $(1^5)$ or inhomogeneous $(21^3)$. In the inhomogeneous case, there are three possibilities for the second 5-class group, namely $\langle 5^7, 891\rangle$, $\langle 5^7, 894\rangle$ and $\langle 5^7, 897\rangle$. In the homogeneous case, the second 5-class group $\langle 5^7, 885\rangle$ is unique. According to the Shafarevich Theorem [20, Th. 6, Eqn. (18′)], whose misprint we have corrected in [15, Th. 5.1, p. 28], these four groups, which possess relation rank $d_2 = 4$, are forbidden as 5-class tower groups for imaginary

**Table 4** The group $G_5^{(2)}M$ of $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $0 < d < 5000$

| No. | Discriminant | | Principalization | | $G_5^{(2)}M$ | $\ell_5 M$ | Invariants | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d$ | Factors | $\varkappa$ | Remark | | | $r_1$ | $\delta_1$ | $r_2$ | $\delta_2$ | Case |
| 1 | 257 | Prime | (660666) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 2 | 457 | Prime | (234156) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 3 | 501 | 3, 167 | (521346) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 4 | 508 | 4, 127 | (653421) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 5 | 509 | Prime | (216453) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 6 | 581 | 7, 83 | (123456) | Identity | $\langle 5^5, 14\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 7 | 629 | 17, 37 | (154326) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 8 | 753 | 3, 251 | (123456) | Identity | $\langle 5^5, 14\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 9 | 764 | 4, 191 | (666066) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 10 | 881 | Prime | (463152) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 11 | 1113 | 3, 7, 53 | (653421) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 12 | 1137 | 3, 379 | (444444) | Constant | $\langle 5^7, 891 \vert 894 \vert 897\rangle$ | $\geq 3$ | 0 | 0 | 0 | 0 | (d) |
| 13 | 1192 | 8, 149 | (463152) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 14 | 1704 | 8, 3, 71 | (653421) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 15 | 1708 | 4, 7, 61 | (404444) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 16 | 1829 | 31, 59 | (216453) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 17 | 1853 | 17, 109 | (550555) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 18 | 1996 | 4, 499 | (613254) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 1 | 1 | 0 | 0 | (e) |
| 19 | 2008 | 8, 251 | (550555) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 20 | 2189 | 11, 199 | (505555) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 21 | 2296 | 8, 7, 41 | (123456) | Identity | $\langle 5^5, 14\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 22 | 2573 | 31, 83 | (613254) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 23 | 2829 | 3, 23, 41 | (123456) | Identity | $\langle 5^5, 14\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 24 | 3121 | Prime | (532416) | Two 2-Cycles | $\langle 5^5, 7\rangle$ | 2 | 1 | 1 | 0 | 0 | (e) |
| 25 | 3129 | 3, 7, 149 | (333303) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 1 | 1 | 0 | 0 | (e) |
| 26 | 3169 | Prime | (444444) | Constant | $\langle 5^7, 891 \vert 894 \vert 897\rangle$ | $\geq 3$ | 0 | 0 | 0 | 0 | (d) |
| 27 | 3253 | Prime | (243651) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 1 | 1 | 0 | 0 | (e) |
| 28 | 4189 | 59, 71 | (243651) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 29 | 4357 | Prime | (000000) | Abelian | $\langle 5^2, 2\rangle$ | 1 | 1 | 0 | 0 | 1 | (a) |
| 30 | 4444 | 4, 11, 101 | (000000) | Abelian | $\langle 5^2, 2\rangle$ | 1 | 1 | 0 | 0 | 1 | (a) |
| 31 | 4461 | 3, 1487 | (653421) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 32 | 4504 | 8, 563 | (444404) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 1 | 1 | 1 | 1 | (c) |
| 33 | 4553 | 29, 157 | (123456) | Identity | $\langle 5^5, 14\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |

**Table 4** continued

| No. | Discriminant | | Principalization | | $G_5^{(2)}M$ | $\ell_5 M$ | Invariants | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d$ | Factors | $\varkappa$ | Remark | | | $r_1$ | $\delta_1$ | $r_2$ | $\delta_2$ | Case |
| 34 | 4697 | 7, 11, 61 | (000000) | tot., non-ab. | $\langle 5^5, 3\rangle \downarrow$ | $\geq 3$ | 0 | 0 | 0 | 0 | (d) |
| 35 | 4709 | 17, 277 | (444444) | Constant | $\langle 5^7, 885\rangle$ | $\geq 3$ | 0 | 0 | 0 | 0 | (d) |
| 36 | 4861 | Prime | (333303) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 37 | 4957 | Prime | (135246) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |

**Table 5** The group $G_5^{(2)}M$ of $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $5000 < d < 10000$

| No. | Discriminant | | Principalization | | $G_5^{(2)}M$ | $\ell_5 M$ | Invariants | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d$ | Factors | $\varkappa$ | Remark | | | $r_1$ | $\delta_1$ | $r_2$ | $\delta_2$ | Case |
| 38 | 5116 | 4, 1279 | (123456) | Identity | $\langle 5^5, 14\rangle$ | 2 | 0 | 0 | 1 | 1 | (f) |
| 39 | 5129 | 23, 223 | (526431) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 40 | 5233 | Prime | (142536) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 41 | 5241 | 3, 1747 | (660666) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 1 | 1 | 0 | 0 | (e) |
| 42 | 5269 | 11, 479 | (222220) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 1 | 1 | 0 | 0 | (e) |
| 43 | 5308 | 4, 1327 | (513462) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 1 | 1 | 0 | 0 | (e) |
| 44 | 5361 | 3, 1787 | (625413) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 45 | 5393 | Prime | (440444) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 46 | 5464 | 8, 683 | (440444) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 47 | 5557 | Prime | (111111) | Constant | $\langle 5^7, 885\rangle$ | $\geq 3$ | 0 | 0 | 0 | 0 | (d) |
| 48 | 5736 | 8, 3, 239 | (123456) | Identity | $\langle 5^5, 14\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 49 | 5989 | 53, 113 | (440444) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 50 | 6072 | 8, 3, 11, 23 | (613254) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 51 | 6073 | Prime | (000000) | tot., non-ab. | $\langle 5^5, 3\rangle \downarrow$ | $\geq 3$ | 0 | 0 | 0 | 0 | (d) |
| 52 | 6113 | Prime | (421653) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 1 | 1 | 0 | 0 | (e) |
| 53 | 6524 | 4, 7, 233 | (513462) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 54 | 6761 | Prime | (123456) | Identity | $\langle 5^5, 14\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 55 | 6949 | Prime | (666066) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 1 | 1 | 1 | 1 | (c) |
| 56 | 6952 | 8, 11, 79 | (220222) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 57 | 7032 | 8, 3, 293 | (213546) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 58 | 7041 | 3, 2347 | (666666) | Constant | $\langle 5^7, 885\rangle$ | $\geq 3$ | 0 | 0 | 0 | 0 | (d) |
| 59 | 7221 | 3, 29, 83 | (444404) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 1 | 1 | 1 | 1 | (c) |
| 60 | 7229 | Prime | (444444) | Constant | $\langle 5^7, 885\rangle$ | $\geq 3$ | 1 | 1 | 1 | 1 | (c) |
| 61 | 7336 | 8, 7, 131 | (606666) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |

**Table 5** continued

| No. | Discriminant | | Principalization | | $G_5^{(2)}M$ | $\ell_5 M$ | Invariants | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d$ | Factors | $\varkappa$ | Remark | | | $r_1$ | $\delta_1$ | $r_2$ | $\delta_2$ | Case |
| 62 | 7361 | 17, 433 | (653421) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 63 | 7489 | Prime | (123456) | Identity | $\langle 5^5, 14\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 64 | 7628 | 4, 1907 | (164253) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 65 | 7656 | 8, 3, 11, 29 | (444404) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 66 | 7752 | 8, 3, 17, 19 | (623145) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 67 | 7833 | 3, 7, 373 | (326154) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 68 | 7996 | 4, 1999 | (022222) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 69 | 8008 | 8, 7, 11, 13 | (625413) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 70 | 8012 | 4, 2003 | (165432) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 71 | 8309 | 7, 1187 | (111110) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 1 | 1 | 0 | 0 | (e) |
| 72 | 8689 | Prime | (002001) | Coclass 4 | $\langle 5^7, 115\rangle \downarrow$ | $\geq 3$ | 1 | 1 | 0 | 0 | (e) |
| 73 | 8789 | 11, 17, 47 | (362451) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 74 | 8877 | 3, 11, 269 | (463152) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 75 | 8972 | 4, 2243 | (362451) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 1 | 1 | (f) |
| 76 | 9013 | Prime | (123456) | Identity | $\langle 5^5, 14\rangle$ | 2 | 0 | 0 | 1 | 1 | (f) |
| 77 | 9052 | 4, 31, 73 | (333303) | Nearly const. | $\langle 5^5, 4\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 78 | 9544 | 8, 1193 | (125364) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 79 | 9564 | 4, 3, 797 | (425136) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 80 | 9573 | 3, 3191 | (216453) | Two 2-cycles | $\langle 5^5, 7\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 81 | 9669 | 3, 11, 293 | (362451) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 1 | 1 | 1 | 1 | (c) |
| 82 | 9752 | 8, 23, 53 | (513462) | 4-Cycle | $\langle 5^5, 11\rangle$ | 2 | 0 | 0 | 0 | 0 | (d) |
| 83 | 9829 | Prime | (123456) | Identity | $\langle 5^5, 14\rangle$ | 2 | 1 | 1 | 0 | 0 | (e) |

cyclic quartic fields with unit rank 1. Therefore, the length of the 5-class tower must be $\ell_5 M \geq 3$ at least, and we conjecture a precise three-stage tower $\ell_5 M = 3$.

The *complete statistics* of the 83 *imaginary* cyclic quartic fields $M$ with $0 < d < 10^4$ is as follows:

- There are 23 (about 28%) cases with $G_5^{(2)} M \simeq \langle 5^5, 11\rangle$, the Schur $\sigma$-group with transfer kernel type a 4-cycle.
- There are 22 (about 27%) cases with $G_5^{(2)} M \simeq \langle 5^5, 4\rangle$.
- There are 16 (about 19%) cases with $G_5^{(2)} M \simeq \langle 5^5, 7\rangle$.
- There are 11 (about 13%) cases with $G_5^{(2)} M \simeq \langle 5^5, 14\rangle$, the Schur $\sigma$-group with transfer kernel type the identity permutation.
- For only 4 cases we have $G_5^{(2)} M \simeq \langle 5^7, 885\rangle$.
- For 2 cases $G_5^{(2)} M \simeq \langle 5^7, 891|894|897\rangle$.
- For 2 cases $G_5^{(2)} M \simeq \langle 5^2, 2\rangle$, the elementary bicyclic 5-group of rank 2.

**Table 6** The group $G_5^{(\infty)}M$ of $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $-100000 < d < 0$

| No. | Discriminant | | Principalization | | $G_5^{(\infty)}M$ | $\ell_5 M$ | Invariants | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d$ | Factors | $\varkappa$ | Type | | | $r_1$ | $\delta_1$ | $r_2$ | $\delta_2$ | Case |
| 1 | −12883 | 13, 991 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 2 | −13147 | Prime | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 3 | −14339 | 13, 1103 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 4 | −15419 | 17, 907 | (000000) | a.1 ↑ | $\langle 5^4, 7\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 5 | −16724 | 4, 37, 113 | (000000) | a.1 ↑ | $\langle 5^4, 7\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 6 | −23336 | 8, 2917 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 7 | −23732 | 4, 17, 349 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 8 | −26743 | 47, 569 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 9 | −27528 | 8, 3, 31, 37 | (003000) | a.2, fixed pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 10 | −27939 | 3, 67, 139 | (000050) | a.2, fixed pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 11 | −28696 | 8, 17, 211 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 12 | −31103 | 19, 1637 | (000000) | a.1 ↑ | $\langle 5^4, 7\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 13 | −35067 | 3, 11689 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 14 | −35839 | Prime | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 15 | −38984 | 8, 11, 443 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 16 | −39947 | 43, 929 | (003000) | a.2, fixed pt. | $\langle 5^4, 8\rangle$ | 2 | 2 | 1 | 1 | 0 | (d) |
| 17 | −40823 | Prime | (000050) | a.2, fixed pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 18 | −42899 | Prime | (000000) | a.1 ↑ | $\langle 5^4, 7\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 19 | −47172 | 4, 3, 3931 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 20 | −52276 | 4, 7, 1867 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 21 | −54347 | Prime | (100000) | a.2, fixed pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 22 | −55667 | Prime | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 23 | −56167 | Prime | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 24 | −58424 | 8, 67, 109 | (000000) | Abelian | $\langle 5^2, 2\rangle$ | 1 | 2 | 0 | 0 | 0 | (a) |
| 25 | −64103 | 13, 4931 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 2 | 1 | 1 | 0 | (d) |
| 26 | −64724 | 4, 11, 1471 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 27 | −67128 | 8, 3, 2797 | (000000) | a.1 ↑ | $\langle 5^4, 7\rangle$ | 2 | 2 | 1 | 1 | 0 | (d) |
| 28 | −69619 | 11, 6329 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 29 | −70763 | 7, 11, 919 | (000000) | a.1 ↑ | $\langle 5^4, 7\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 30 | −74019 | 3, 11, 2243 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 31 | −75103 | 7, 10729 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 32 | −75892 | 4, 18973 | (100000) | a.2, fixed pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 33 | −78747 | 3, 26249 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 34 | −83636 | 4, 7, 29, 103 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 35 | −86404 | 4, 21601 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 36 | −91127 | Prime | (000400) | a.2, fixed pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 37 | −92219 | Prime | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 38 | −99428 | 4, 7, 53, 67 | (003000) | a.2, fixed pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |

**Table 7** The group $G_5^{(\infty)} M$ of $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $-175000 < d < -100000$

| No. | Discriminant | | Principalization | | $G_5^{(\infty)} M$ | $\ell_5 M$ | Invariants | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d$ | Factors | $\varkappa$ | Type | | | $r_1$ | $\delta_1$ | $r_2$ | $\delta_2$ | Case |
| 39 | −100708 | 4, 17, 1481 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 40 | −101011 | 83, 1217 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 41 | −101784 | 8, 3, 4241 | (003000) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 42 | −104503 | 7, 14929 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 2 | 1 | 1 | 0 | (d) |
| 43 | −105431 | 19, 31, 179 | (000400) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 44 | −105784 | 8, 7, 1889 | (000000) | a.1 ↑ | $\langle 5^4, 7\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 45 | −107791 | Prime | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 46 | −110479 | Prime | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 2 | 1 | (e) |
| 47 | −114303 | 3, 7, 5443 | (263415) | 4-cycle | $\langle 5^5, 11\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 48 | −114679 | Prime | (000006) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 49 | −115912 | 8, 14489 | (000000) | Abelian | $\langle 5^2, 2\rangle$ | 1 | 2 | 0 | 0 | 0 | (a) |
| 50 | −119191 | Prime | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 2 | 1 | 1 | 0 | (d) |
| 51 | −123028 | 4, 30757 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 52 | −124099 | 193, 643 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 53 | −125547 | 3, 41849 | (003000) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 54 | −127259 | 11, 23, 503 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 55 | −127519 | 7, 18217 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 56 | −133188 | 4, 3, 11, 1009 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 57 | −134392 | 8, 107, 157 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 58 | −136311 | 3, 7, 6491 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 59 | −139703 | Prime | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 60 | −140232 | 8, 3, 5843 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 61 | −142904 | 8, 17863 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 62 | −145007 | Prime | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 63 | −145668 | 4, 3, 61, 199 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 64 | −148004 | 4, 163, 227 | (003000) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 65 | −148507 | 97, 1531 | (000000) | Abelian | $\langle 5^2, 2\rangle$ | 1 | 2 | 0 | 0 | 0 | (a) |
| 66 | −151879 | 7, 13, 1669 | (000000) | Abelian | $\langle 5^2, 2\rangle$ | 1 | 2 | 0 | 0 | 0 | (a) |
| 67 | −154408 | 8, 19301 | (000000) | Abelian | $\langle 5^2, 2\rangle$ | 1 | 2 | 0 | 0 | 0 | (a) |
| 68 | −155603 | 7, 22229 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 69 | −157028 | 4, 37, 1061 | (003000) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 70 | −157031 | 7, 22433 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 71 | −159679 | 13, 71, 173 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 72 | −160571 | 211, 761 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 73 | −163427 | 11, 83, 179 | (000050) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 74 | −164116 | 4, 89, 461 | (000006) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 75 | −165364 | 4, 41341 | (000400) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 76 | −169752 | 8, 3, 11, 643 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |

**Table 8** The group $G_5^{(\infty)}M$ of $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $-200000 < d < -175000$

| No. | Discriminant | | Principalization | | $G_5^{(\infty)}M$ | $\ell_5 M$ | Invariants | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $d$ | Factors | $\varkappa$ | Type | | | $r_1$ | $\delta_1$ | $r_2$ | $\delta_2$ | Case |
| 77 | −175076 | 4, 11, 23, 173 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 78 | −176459 | Prime | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 79 | −177428 | 4, 44357 | (100000) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 80 | −180583 | 13, 29, 479 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 81 | −181847 | 43, 4229 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 82 | −182968 | 8, 22871 | (000050) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 83 | −185883 | 3, 61961 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 84 | −186187 | Prime | (000400) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 85 | −186271 | Prime | (000050) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 86 | −190387 | Prime | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 87 | −193483 | 191, 1013 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 88 | −193571 | 7, 27653 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 89 | −194487 | 3, 241, 269 | (000000) | a.1↑ | $\langle 5^4, 7\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 90 | −196648 | 8, 47, 523 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 91 | −196707 | 3, 7, 17, 19, 29 | (000050) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 92 | −197752 | 8, 19, 1301 | (000000) | a.1 | $\langle 5^3, 3\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |
| 93 | −199947 | 3, 11, 73, 83 | (000400) | a.2, fix pt. | $\langle 5^4, 8\rangle$ | 2 | 1 | 0 | 1 | 0 | (c) |

- For 2 cases $G_5^{(2)}M$ is a descendant of $\langle 5^5, 3\rangle$ indicated by the symbol ↓.
- The last three groups have the biggest order $5^9$ and coclass 4.
  They posses relation rank $d_2 = 5$, which clearly enforces $\ell_5 M \geq 3$ by the Shafarevich Theorem. Again, we conjecture the equality $\ell_5 M = 3$.

Furthermore, we point out that the groups $\langle 5^5, 4\rangle$ and $\langle 5^5, 7\rangle$ with relation rank $d_2 = 3$ are not strong $\sigma$-groups in the sense of Schoof [19]. They are forbidden as 5-class tower groups for any quadratic field, imaginary or real. However, they are admissible for our imaginary cyclic quartic fields $M$ with unit rank 1, since the subfield $k_0^+ = \mathbb{Q}(\sqrt{5})$ also possesses unit rank 1, and so a strong $\sigma$-group is not required.

## 5.2 Real cyclic quartic fields *M* with *d* < 0

Table 6, resp. Table 7, resp. Table 8, shows the factorized fundamental discriminant $d$ of the dual quadratic field $k_1$, the 5-principalization type $\varkappa = \varkappa(M)$, the 5-class tower group $G_5^{(\infty)}M$, the length $\ell_5 M$ of the 5-class tower, the 5-class ranks $r_1 := \varrho_5(k_1), r_2 := \varrho_5(k_2)$, the invariants $\delta_1, \delta_2$, and the case in Proposition 3.3 for the 38, resp. 38, resp. 17, cyclic quartic fields $M = \mathbb{Q}\left((\zeta_5 - \zeta_5^{-1})\sqrt{d}\right)$ with $-100000 < d < 0$, resp. $-175000 < d < -100000$, resp. $-200000 < d < -175000$.

The *complete statistics* of the 93 *real* cyclic quartic fields $M$ with $-2 \cdot 10^5 < d < 0$ is as follows:

- There are 56 (about 60%) cases with $\mathrm{G}_5^{(\infty)}M \simeq \langle 5^3, 3\rangle$ the extra special 5-group of exponent 5.
- There are 23 (about25%) cases with $\mathrm{G}_5^{(\infty)}M \simeq \langle 5^4, 8\rangle$ having a transfer kernel type with fixed point.
- There are 8 (about 9%) cases with $\mathrm{G}_5^{(\infty)}M \simeq \langle 5^4, 7\rangle$ having total transfer kernels exclusively.
- For only 5 cases we have $\mathrm{G}_5^{(\infty)}M \simeq \langle 5^2, 2\rangle$ the elementary bicyclic 5-group of rank 2.
- For a unique case $\mathrm{G}_5^{(\infty)}M \simeq \langle 5^5, 11\rangle$, Schur $\sigma$-group with transfer kernel type a 4-cycle.

The 5-class tower of $M$ possesses length $\ell_5 M = 1$ for the abelian $\mathrm{G}_5^{(\infty)}M \simeq \langle 5^2, 2\rangle$, and $\ell_5 M = 2$ in all other cases.

According to the Shafarevich Theorem [20, Thm. 6, Eqn. (18′)], whose misprint we have corrected in [15, Thm. 5.1, p. 28], the mainline groups $\langle 5^3, 3\rangle$ and $\langle 5^4, 7\rangle$ with relation rank $d_2 = 4$ are forbidden as 5-class tower groups for real quadratic fields with unit rank 1, but they are admissible for real cyclic quartic fields, which have bigger unit rank 3.

### 5.3 The Galois action confirmed

All numerical results in the Tables 4, 5, 6, 7 and 8 are in perfect accordance with Theorems 2.2, 2.3 and Corollary 2.1. A rigorous check with the computational algebra system MAGMA [2,12] proves that only the two terminal Schur $\sigma$-groups $\langle 5^5, 11\rangle$, $\langle 5^5, 14\rangle$ and five other capable top vertices $\langle 5^5, 3\rangle$, $\langle 5^5, 4\rangle$, $\langle 5^5, 5\rangle$, $\langle 5^5, 6\rangle$, $\langle 5^5, 7\rangle$ in the stem of Hall's isoclinism family $\Phi_6$, and the abelian root $\langle 5^2, 2\rangle$, together with their descendants [16], are admissible for $\mathrm{G}_5^{(2)}M$ of any cyclic quartic field $M$, as drawn in Fig. 4.

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