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Proving the Conjecture of Arnold Scholz

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Towers of p-Class Fields over Algebraic Number Fields

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Figure 1. My Website Devoted to the Memory of Arnold Scholz

Centennial 2004

1904 - 2004

Dem Genius

Arnold Scholz

zum Gedächtnis

*** 24. 12. 1904, Berlin + 01. 02. 1942, Flensburg**

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Centennial 2004
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Kurz-Biographie von Arnold Scholz:

1904: geboren in **Berlin-Charlottenburg**
1911 - 1915: Vorschule
1915 - 1923: Kaiserin Augusta Gymnasium in Charlottenburg
1923 - 1928: Studium der Mathematik, Philosophie und Musikwissenschaft an der **Universität Berlin**
1927: ein Semester bei **Ph. Furtwängler** an der **Universität Wien**
1928: Promotio magna cum laude ("spondeo et polliceor") bei **Issai Schur**
1928 - 1930: Assistent an der Berliner Universität
1930 - 1935: Privat-Dozent in **Freiburg** (im Breisgau)
1935 - 1940: Lehrauftrag und Mitglied der Prüfungskommission in **Kiel**
1940: Kriegsdienst
1941: Mathematiklehrer an der **Marine-Akademie Flensburg-Mürwick**
1942: gestorben in Flensburg
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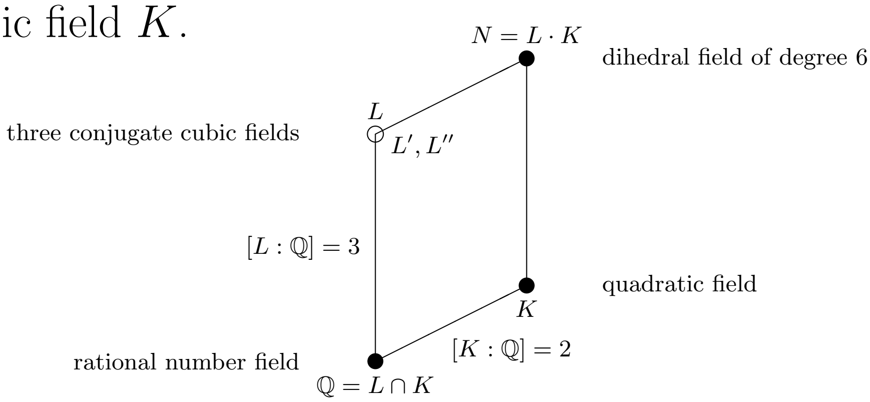
1. Historical Origin of Scholz's Conjecture

Let L be a non-cyclic *totally real cubic field*. Then L is non-Galois over the rational number field $\mathbb{Q}$ with two conjugate fields L' and L''. The Galois closure N of L is a totally real dihedral field of degree 6, which contains a unique real quadratic field K.



In **1930, Helmut Hasse** [4] determined the *discriminants* of L [4, pp. 567 and 575] and N [4, p. 566], in dependence on the discriminant of K and on the class field theoretic *conductor* $f = f(N/K)$ of the cyclic cubic, and thus abelian, relative extension N/K:

$$d_L = f^2 \cdot d_K, \quad \text{and} \quad d_N = f^4 \cdot d_K^3. \tag{1.1}$$

Three years later, in **1933, Arnold Scholz** [19, p. 216] determined the *relation*

$$h_N = \frac{I}{9} \cdot h_L^2 \cdot h_K \tag{1.2}$$

between the class numbers of the fields N, L and K, in dependence on the *index of subfield units*, $I = (U_N : U_0) = 3^E$, where $U_0 = \langle U_K, U_L, U_{L'}, U_{L''} \rangle$ and $E \in \{0, 1, 2\}$.

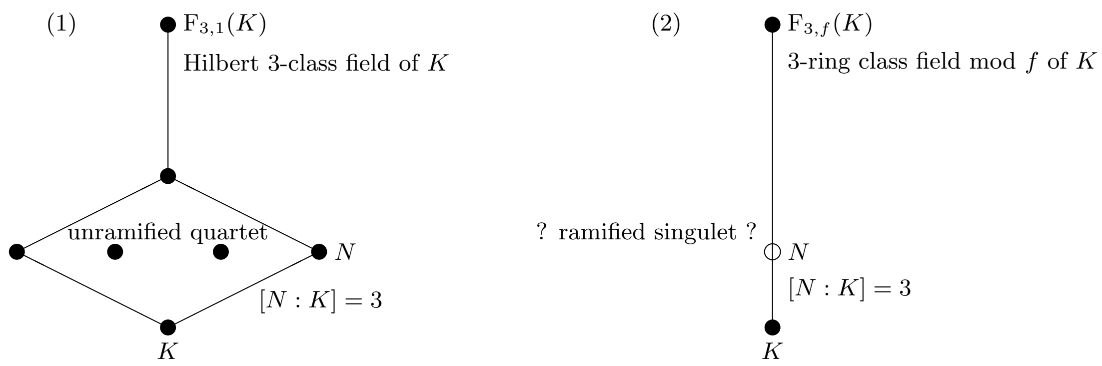
Note that $E = 0$, resp. $I = 1$, is the *distinguished case* where the unit group U_N of the normal field N is entirely generated by all proper subfield units, that is $U_N = U_0$.

Scholz was able to give explicit numerical examples [19, p. 216], for $E = 1$ ($d_L = 229$), resp. $E = 2$ ($d_L = 148$), but not for $E = 0$, and he formulated the following hypothesis.

Conjecture 1.1. (**The Conjecture of Scholz, 1933**) There should exist non-Galois totally real cubic fields L whose Galois closure N is either

(1) *unramified*, with conductor $f = 1$ over some real quadratic field K with 3-class rank $\varrho_3(K) = 2$ whose complete 3-elementary class group capitulates in N, and such that $U_N = U_0$ [19, p. 219], or

(2) *ramified*, with conductor $f > 1$ over some real quadratic field K such that also $U_N = U_0$ [19, p. 221] (here, Scholz calls N a *ring class field* over K, by abuse of language).



2. Complete Proof of Scholz's Conjecture

We point out that, in the *unramified* situation $f = 1$, $d_L = d_K$ is a quadratic fundamental discriminant, and $d_N = d_K^3$ is a perfect cube, according to formula (1.1). In this unramified case, the verification of Conjecture 1.1 can be obtained from a more general theorem, since any real quadratic field K with 3-class rank $\varrho_3(K) = 2$ possesses a multiplet of four unramified cyclic cubic extensions $N_1, \ldots, N_4$, that is a *quartet* of absolutely dihedral fields of degree 6 [14] with non-Galois totally real subfields $L_1, \ldots, L_4$, each of them selected among three conjugate fields.

For such a quartet, Chang and Foote [2] introduced the concept of the *capitulation number* $0 \le \nu(K) \le 4$, defined as the number of those members of the quartet in which the complete 3-elementary class group of K capitulates. For this number $\nu(K)$, the following theorem holds.

Theorem 2.1. *For each value* $0 \le \nu \le 4$*, there exists a real quadratic field* K *with* 3*-class rank* $\varrho_3(K) = 2$ *and capitulation number* $\nu(K) = \nu$*. It is even possible to restrict the claim to fields with elementary* 3*-class group of type* $\mathrm{Cl}_3(K) \simeq C_3 \times C_3$*.*

Proof. From the viewpoint of finite p-group theory, this theorem is a proven statement about the possible *transfer kernel types* of finite metabelian 3-groups G with abelianization $G/G' \simeq (3,3)$ applied to the second 3-class group $G := \mathrm{Gal}(F_{3,1}^2(K)/K)$ of K [14]. □

Proof. (Alternative direct proof.) However, it is easier to give explicit numerical paradigms for each value of $\nu(K)$. We have the following minimal occurrences:

$\nu(K) = 4$ for $d_K = 62\,501$, $\nu(K) = 3$ for $d_K = 32\,009$,
$\nu(K) = 2$ for $d_K = 710\,652$, $\nu(K) = 1$ for $d_K = 534\,824$,
$\nu(K) = 0$ for $d_K = 214\,712$,

which have been computed by ourselves in [14]. □

Remark 2.1. We have the priority of discovering the first examples of real quadratic fields K with $\nu(K) \in \{0, 1, 2\}$ in [14]. However, the first examples of real quadratic fields K with $\nu(K) \in \{3, 4\}$ are due to Heider and Schmithals [5], who performed a mainframe computation on the CDC Cyber of the University of Cologne, and so the following corollary is proven since 1982 already.

Corollary 2.1. *(Verification of Conjecture 1.1, (1) for* ***unramified*** *extensions)*
There exist non-Galois totally real cubic fields L whose Galois closure N is unramified, with conductor $f = 1$, over a real quadratic field K with 3-class rank $\varrho_3(K) = 2$ whose complete 3-elementary class group capitulates in N, and which therefore has $U_N = U_0$. The minimal discriminant of such a field L is $d_L = 32\,009$ (three of four fields, 1982 [5]*,* 49 *years after* 1933 [19]*).*

Proof. It suffices to take a real quadratic field K with $1 \leq \nu(K) \leq 4$ in Theorem 2.1. In view of the minimal discriminant, we select $\nu(K) = 3$ and obtain $U_N = U_0$ for three of four fields with $d_L = d_K = 32\,009$ (unramified quartet). □

Concerning the *ramified* situation $f > 1$ in Conjecture 1.1 (2), Scholz does not explicitly impose any conditions on the underlying real quadratic field K. We suppose that he also tacitly assumed a real quadratic field K with 3-class rank $\varrho_3(K) = 2$.

However, more recent extensions of the theory of dihedral fields by means of *differential principal factorizations* and *Galois cohomology* revealed that for $U_N = U_0$ no constraints on the p-class rank $\varrho_p(K)$ are required. In 1975, Nicole Moser [18] used the *Galois cohomology* $\mathrm{H}^0(G, U_N) \simeq U_K/\mathrm{N}_{N/K}(U_N)$ of the unit group U_N of the normal closure N as a module over $G = \mathrm{Gal}(N/K)$ to establish a *fine structure* with five possible types $\alpha, \beta, \gamma, \delta, \varepsilon$ on the *coarse* classification by three possible values of the index $(U_N : U_0)$ of subfield units:

$(U_N : U_0) = 1 \iff$ type α with $(U_K : \mathrm{N}_{N/K}(U_N)) = 3$,
$(U_N : U_0) = 3 \iff$ type β with $(U_K : \mathrm{N}_{N/K}(U_N)) = 3$ or
type δ with $(U_K : \mathrm{N}_{N/K}(U_N)) = 1$,
$(U_N : U_0) = 9 \iff$ type γ with $(U_K : \mathrm{N}_{N/K}(U_N)) = 3$ or
type ε with $(U_K : \mathrm{N}_{N/K}(U_N)) = 1$.

Thus, Moser's refinement does not illuminate the situation $U_N = U_0$ ($\iff$ type α) of Scholz's conjecture more closely. Meanwhile, Barrucand and Cohn [1] had coined the notion of *differential principal factorization* (DPF) for pure cubic fields. In 1991, we generalized the theory of DPFs for dihedral fields of both signatures [10], and we obtained a *hyperfine structure* by splitting Moser's types further according to the $\mathbb{F}_p$-dimensions C of the capitulation kernel $\ker(T_{K,N})$ and R of the space of relative DPFs of N/K.

In particular, type α with $U_N = U_0$ splits into three subtypes:
type $\alpha_1 \Longleftrightarrow C = 2, R = 0$, which implies $\varrho_p(K) \geq 2$,
type $\alpha_2 \Longleftrightarrow C = 1, R = 1$, which implies $\varrho_p(K) \geq 1$ and a split prime divisor of f ($s \geq 1$),
type $\alpha_3 \Longleftrightarrow C = 0, R = 2$, which is compatible with any $\varrho_p(K) \geq 0$, but requires $s \geq 2$.
Consequently, we were led to the following refinement of Conjecture 1.1 (2).

Conjecture 2.1. (**Conjecture of D.C. Mayer, 1991**)
Non-Galois totally real cubic fields L whose Galois closure N is *ramified*, with conductor $f > 1$, over some real quadratic field K, and is of type α, with $U_N = U_0$, should exist for each of the following three situations:
(2.1) type α_1 with $\dim_{\mathbb{F}_3}(\ker(T_{K,N})) = \varrho_3(K) = 2$, $s = 0$,
(2.2) type α_2 with $\dim_{\mathbb{F}_3}(\ker(T_{K,N})) = \varrho_3(K) = 1$, $s = 1$,
(2.3) type α_3 with $\dim_{\mathbb{F}_3}(\ker(T_{K,N})) = \varrho_3(K) = 0$, $s = 2$,
where $T_{K,N} : \mathrm{Cl}_3(K) \to \mathrm{Cl}_3(N)$, $\mathfrak{a} \cdot \mathcal{P}_K \mapsto (\mathfrak{a}\mathcal{O}_N) \cdot \mathcal{P}_N$, denotes the *transfer homomorphism* of 3-classes from K to N, and s counts the prime divisors of the conductor f which *split* in K.

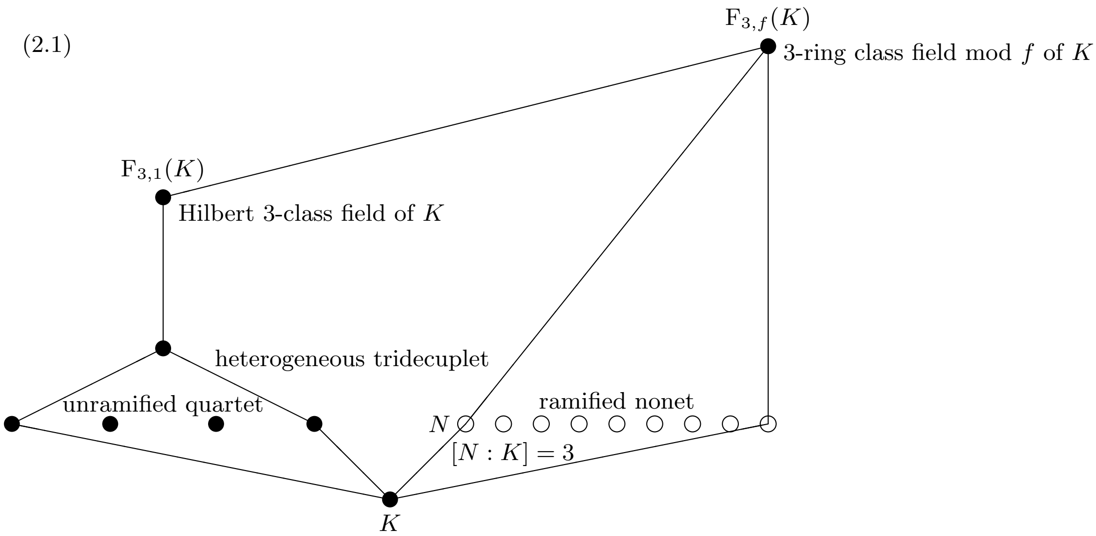
Theorem 2.2. *(Verification of Conjecture 2.1, (2.3))*
There exist non-Galois totally real cubic fields L whose Galois closure N is ramified, with conductor $f > 1$ divisible by two prime divisors which split in K, over a real quadratic field K with 3-class rank $\varrho_3(K) = 0$, without capitulation in N, but which nevertheless has $U_N = U_0$. The minimal discriminant of such a field L is $d_L = 146\,853 = (7 \cdot 9)^2 \cdot 37$ (singulet, 1991 [11], 58 years after 1933 [19]).

Proof. This was proved in the numerical supplement [11] of our paper [10] by computing a gapless list of all 10 015 totally real cubic fields L with discriminants $d_L < 200\,000$ on the AMDAHL mainframe of the University of Manitoba. There occurred the minimal discriminant $d_L = 146\,853 = f^2 \cdot d_K$ with $d_K = 37$ and conductor $f = 63 = 3^2 \cdot 7$ divisible by two primes which both split in K, i.e. $s = 2$. This is a necessary requirement for a two-dimensional relative principal factorization with $R = 2$ and is unique up to $d_L < 200\,000$. Only a single field L has discriminant $d_L = 146\,853$. □

Our discovery of the truth of Theorem 2.2 with the aid of the list [11] was a random hit without explicit intention to find a verification of Scholz's conjecture. Unfortunately, [11] does not contain examples of the *unique missing* DPF type α_2. It required more than 25 years until we focused on an attack against this lack of information. In contrast to the techniques of [11], we did not use the Voronoi algorithm [20] after cumbersome preparation of generating polynomials for totally real cubic fields, but rather the *class field theory routines* of Magma [8] for a direct generation of the fields as subfields of 3-ray class fields modulo conductors $f > 1$.

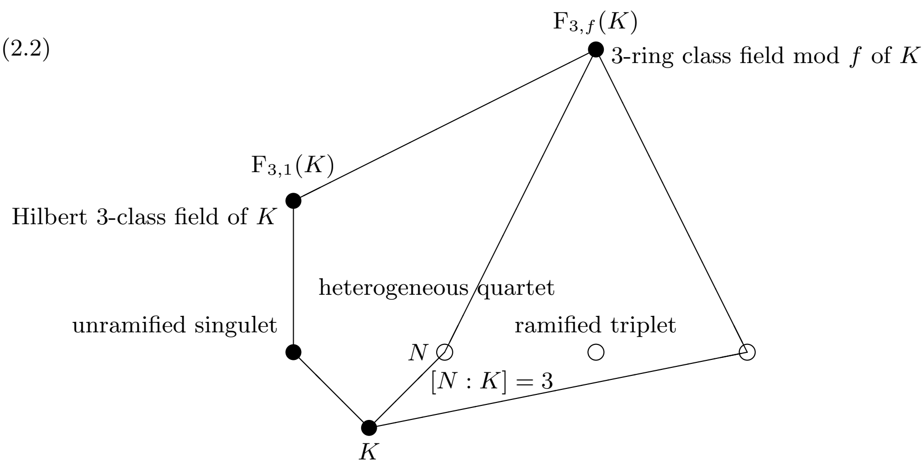
Theorem 2.3. *(Verification of Conjecture 2.1, (2.1)) There exist non-Galois totally real cubic fields L whose Galois closure N is ramified, with conductor $f > 1$ divisible only by prime divisors which do not split in K, over a real quadratic field K with 3-class rank $\varrho_3(K) = 2$, with complete capitulation of the elementary 3-class group in N, and thus with $U_N = U_0$. The minimal discriminant of such a field L is $d_L = 18\,251\,060 = 2^2 \cdot 4\,562\,765$ (five of nine fields, 23 November 2017, 84 years after 1933 [19]).*

(2.1)



Theorem 2.4. *(Verification of Conjecture 2.1, (2.2)) There exist non-Galois totally real cubic fields L whose Galois closure N is ramified, with conductor $f > 1$ divisible by a single prime divisor that splits in K, i.e. $s = 1$ over a real quadratic field K with 3-class rank $\varrho_3(K) = 1$, with complete capitulation of the elementary 3-class group in N, but nevertheless with $U_N = U_0$. The minimal discriminant of such a field L is $d_L = 966\,397 = 19^2 \cdot 2\,677$ (two of three, 19 November 2017, 84 years after 1933 [19]).*

(2.2)



Proof. The proof of Theorem 2.3 and Theorem 2.4 is conducted in the following sections on real quadratic base fields with 3-class rank 1 and 2. □

2.1. **Real quadratic base fields with 3-class rank 1.** In Table 1, we present the results of our search for the *minimal discriminant* d_L, resp. d_N, of a non-Galois totally real cubic field L, resp. its normal closure N, with *differential principal factorization type* α_2. The unramified component is a *singulet* which must be of DPF type δ_1. For each member of the ramified *triplet* the DPF types $\alpha_2, \beta_1, \beta_2, \delta_1, \delta_2, \varepsilon$ would be possible, but only the types $\alpha_2, \delta_1, \delta_2$ occur actually.

The desired minimum is clearly given by $d_L = 19^2 \cdot 2\,677 = 966\,397$ with two occurrences of ramified extensions with DPF type α_2.

Table 1. Heterogeneous quartets of dihedral fields with a splitting prime f

f	d_K	$d_L = f^2 \cdot d_K$	unramified component δ_1	ramified components α_2	δ_1	δ_2
3^2	14 197	1 149 957	1	3	0	0
7	21 781	1 067 269	1	2	1	0
13	9 749	1 647 581	1	2	0	1
19	2 677	966 397	1	2	1	0
31	3 877	3 725 797	1	2	0	1
37	5 477	7 498 013	1	1	0	2
43	4 933	9 121 117	1	3	0	0
61	3 981	14 813 301	1	3	0	0
67	4 493	20 169 077	1	2	0	1
73	10 733	57 196 157	1	3	0	0

2.2. **Real quadratic base fields with 3-class rank 2.** In this situation, the unramified *quartet* is non-trivial, since the two DPF types α_1 and δ_1 are possible. These quartets have been studied thoroughly in [14], and in the Tables 2 and 3, we use the corresponding notation for *capitulation types*.

In Table 2, we present the results of the crucial search for the *minimal discriminant* d_L, resp. d_N, of a non-Galois totally real cubic field L, resp. its normal closure N, with *differential principal factorization type* α_1 such that N/K is a *ramified* extension of a real quadratic field K with 3-class rank $\varrho_3 = 2$. We tried to fix the minimal possible conductor $f > 1$, namely $f = 2$. This experiment was motivated by the fact that the conductor f enters the expression $d_L = f^2 \cdot d_K$ in its second power, whereas the quadratic discriminant d_K enters linearly. Consequently, the probability to find the minimum of d_L is higher for small f than for small d_K.

The table is ordered by increasing quadratic fundamental discriminants d_K and gives $d_L = 2^2 \cdot d_K$ and the *extended Artin pattern* of the *heterogeneous tridecuplet* of cyclic cubic relative extensions N/K consisting of an *unramified quartet* $(N_{1,1}, \ldots, N_{1,4})$ with conductor $f' = 1$ and a *ramified nonet* $(N_{2,1}, \ldots, N_{2,9})$ with conductor $f = 2$, grouped by the possible two, resp. four, DPF types. Capitulation kernels $\varkappa$ are abbreviated by digits, 0 for two-dimensional and $1, \ldots, 4$ for one-dimensional principalization, and an asterisk $*$ for a trivial kernel. Transfer targets τ are abbreviated by logarithmic abelian type invariants of 3-class groups. Symbolic exponents always denote repetition.

The desired minimum is given by $d_L = 4 \cdot 4\,562\,765 = 18\,251\,060$ with five occurrences of ramified extensions with DPF type α_1. Generally, there is an abundance of ramified extensions with two-dimensional capitulation kernel: at least three and at most all nine of a nonet.

TABLE 2. Heterogeneous tridecuplets of dihedral fields with $f = 2$

		unramified components				ramified components							
		α_1		δ_1		α_1		β_1		δ_1		ε	
d_K	Type	$\varkappa$	τ	$\varkappa$	τ	$\varkappa$	τ	$\varkappa$	τ	$\varkappa$	τ	$\varkappa$	τ
4 562 765	a.3*	0^3	$(1^2)^3$	1	1^3	0^5	$2^21^2, (1^4)^4$	1	1^5	14^2	$(21^3)^3$		
7 339 397	a.3*	0^3	$(1^2)^3$	1	1^3	0^7	$(1^4)^7$	2	21^3	1	21^3		
7 601 461	a.3	0^3	$(1^2)^3$	1	21	0^6	$2^21^2, (1^4)^5$			234	$(21^3)^3$		
7 657 037	a.3	0^3	$(1^2)^3$	1	21	0^6	$(1^4)^6$	1	21^3	12	$1^5, 21^3$		
7 736 749	a.3*	0^3	$(1^2)^3$	1	1^3	0^7	$(1^4)^7$			4^2	$(21^3)^2$		
8 102 053	a.3*	0^3	$(1^2)^3$	1	1^3	0^7	$(1^4)^7$			23	$1^5, 21^3$		
9 182 229	a.2	0^3	$(1^2)^3$	4	21	0^8	$2^21^2, (1^4)^7$			2	21^3		
9 500 453	a.3	0^3	$(1^2)^3$	1	21	0^8	$2^21^2, (1^4)^7$			3	21^3		
9 533 357	a.3	0^3	$(1^2)^3$	1	21	0^6	$(1^4)^6$	1	21^3	23	$(21^3)^2$		
11 003 845	a.3	0^3	$(1^2)^3$	1	21	0^4	$(1^4)^4$			12^24^2	$1^5, (21^3)^4$		
12 071 253	a.3	0^3	$(1^2)^3$	1	21	0^7	$(1^4)^7$	3	21^3	2	21^3		
14 266 853	a.3	0^3	$(1^2)^3$	1	21	0^8	$2^21^2, (1^4)^7$			4	21^3		
14 308 421	a.3*	0^3	$(1^2)^3$	1	1^3	0^4	$(1^4)^4$			1^2234	$2^31, (21^3)^3, 1^5$		
14 315 765	a.3	0^3	$(1^2)^3$	1	21	0^7	$(1^4)^7$			23	$(21^3)^2$		
14 395 013	a.3*	0^3	$(1^2)^3$	1	1^3	0^6	$(1^4)^6$	1	21^3	23	$(21^3)^2$		
15 131 149	D.10			2414	$(21)^3, 1^3$	0^7	$(1^4)^7$	1	21^3	1	21^3		
16 385 741	a.3*	0^3	$(1^2)^3$	1	1^3	0^4	$(1^4)^4$			23^24	$(21^3)^4$	$*$	32^21

Table 3 shows analogous results for the conductor $f = 5$, that is, $d_L = 5^2 \cdot d_K$. The minimum $d_L = 25 \cdot 1\,049\,512 = 26\,237\,800$ is clearly beaten by the minimum $4 \cdot 4\,562\,765 = 18\,251\,060$ in Table 2.

TABLE 3. Heterogeneous tridecuplets of dihedral fields with $f = 5$

		unramified components				ramified components							
		α_1		δ_1		α_1		β_1		δ_1		ε	
d_K	Type	$\varkappa$	τ	$\varkappa$	τ	$\varkappa$	τ	$\varkappa$	τ	$\varkappa$	τ	$\varkappa$	τ
1 049 512	a.3	0^3	$(1^2)^3$	1	21	0^4	$(1^4)^4$			234^3	$(21^3)^5$		
2 461 537	a.2	0^3	$(1^2)^3$	4	21	0^7	$(1^4)^7$			12	$(21^3)^2$		
2 811 613	a.3*	0^3	$(1^2)^3$	1	1^3	0^5	$2^21^2, (1^4)^4$	2	21^3	123	$(21^3)^3$		
3 091 133	a.3	0^3	$(1^2)^3$	1	21	0^4	$(1^4)^4$	4	21^3	1^32	$(21^3)^4$		
5 858 753	G.19			2143	$(21)^4$	0^7	$(2^21^2)^3, (1^4)^4$			3	21^3	$*$	21^4
6 036 188	D.10			3431	$1^3, (21)^3$	0^8	$(1^4)^8$					$*$	2^21^2

Since we know a small candidate $d_L = 966\,397$ for the minimal discriminant, and since the smallest quadratic discriminant with $\varrho_3(K) = 1$ is $d_K = 229$, we only have to investigate prime and composite conductors $f = \sqrt{\frac{d_L}{d_K}}$ with $s \geq 1$ and

$$f \leq \sqrt{\frac{966\,397}{229}} \approx \sqrt{4220} \approx 64.9,$$

which are divisible by a split prime, that is,

$$f \in \{7, 9 = 3^2, 13, 14 = 2\cdot 7, 18 = 2\cdot 3^2, 19, 21 = 3\cdot 7, 26 = 2\cdot 13, 31, 35 = 5\cdot 7, 37, 38 = 2\cdot 19, 39 = 3\cdot 13, 42 = 2\cdot 3\cdot 7, 43, 45 = 5\cdot 3^2, 57 = 3\cdot 19, 61, 62 = 2\cdot 31, 63 = 7\cdot 3^2\}.$$

Table 4. Heterogeneous quartets of dihedral fields with conductor f

f	condition	d_K	$d_L = f^2 \cdot d_K$	unramified component δ_1	ramified components α_2	δ_1	δ_2
7		21 781	1 067 269	1	2	1	0
3^2	$d_K \equiv 1\,(3)$	14 197	1 149 957	1	3	0	0
13		9 749	1 647 581	1	2	0	1
19		2 677	966 397	1	2	1	0
31		3 877	3 725 797	1	2	0	1
37		5 477	7 498 013	1	1	0	2
43		4 933	9 121 117	1	3	0	0
61		3 981	14 813 301	1	3	0	0
$2\cdot 7$		6 997	1 371 412	1	3	0	0
$2\cdot 3^2$	$d_K \equiv 1\,(3)$	16 141	5 229 684	1	3	0	0
$3\cdot 7$	$d_K \equiv 3\,(9)$	28 137	12 408 417	1	3	0	0
$3\cdot 7$	$d_K \equiv 6\,(9)$	57 516	25 364 556	1	3	0	0
$2\cdot 13$		21 557	14 572 532	1	3	0	0
$5\cdot 7$		14 457	17 709 825	1	3	0	0
$2\cdot 19$		13 765	19 876 660	1	3	0	0
$3\cdot 13$	$d_K \equiv 3\,(9)$	51 528	78 374 088	1	3	0	0
$3\cdot 13$	$d_K \equiv 6\,(9)$	37 176	56 544 696	1	3	0	0
$5\cdot 3^2$	$d_K \equiv 1\,(3)$	24 952	50 527 800	1	1	1	1
$3\cdot 19$	$d_K \equiv 3\,(9)$	24 393	79 252 857	1	3	0	0
$3\cdot 19$	$d_K \equiv 6\,(9)$	39 417	128 065 833	1	3	0	0
$2\cdot 31$		7 573	29 110 612	1	3	0	0
$7\cdot 3^2$	$d_K \equiv 1\,(3)$	2 941	11 672 829	1	3	0	0
$7\cdot 3^2$	$d_K \equiv 2\,(3)$	23 993	95 228 217	1	3	0	0

The result of the investigations is summarized in Table 4, which clearly shows that $d_L =$ **966 397**, for $d_K = 2\,677$ and splitting prime conductor $f = 19$ bigger than the conductor $f = 1$ of unramified extensions N/K, is the desired **minimal discriminant** of a totally real cubic field with ramified extension N/K, DPF type α_2 and $U_N = U_0$. The information has been computed with class field theoretic routines of Magma [8].

Since we know a small candidate $d_L = 18\,251\,060$ for the minimal discriminant, and since the smallest quadratic discriminant with $\varrho_3(K) = 2$ is $d_K = 32\,009$, we only have to investigate prime and composite conductors $f = \sqrt{\frac{d_L}{d_K}}$ with

$$f \le \sqrt{\frac{18\,251\,060}{32\,009}} \approx \sqrt{570.2} \approx 23.9,$$

that is,

$$f \in \{2, 3, 5, 6 = 2 \cdot 3, 7, 9 = 3^2, 10 = 2 \cdot 5, 11, 13, 14 = 2 \cdot 7, \\ 15 = 3 \cdot 5, 17, 18 = 2 \cdot 3^2, 19, 21 = 3 \cdot 7, 22 = 2 \cdot 11, 23\}.$$

Table 5. Heterogeneous tridecuplets of dihedral fields with conductor f

f	condition	d_K	$d_L = f^2 \cdot d_K$	unramified components α_1	δ_1	ramified components α_1	β_1	δ_1	ε
2		4 562 765	18 251 060	3	1	5	1	3	0
3	$d_K \equiv 3\,(9)$	9 964 821	89 683 389	3	1	4	0	4	1
5		1 049 512	26 237 800	3	1	4	0	5	0
7		966 053	47 336 597	3	1	4	0	4	1
3^2	$d_K \equiv 1\,(3)$	1 482 568	120 088 008	3	1	5	1	2	1
3^2	$d_K \equiv 2\,(3)$	2 515 388	203 746 428	3	1	6	1	2	0
3^2	$d_K \equiv 6\,(9)$	621 429	50 335 749	3	1	6	0	3	0
11		476 152	57 614 392	3	1	7	0	2	0
13		1 122 573	189 714 837	3	1	7	0	2	0
17		665 832	192 425 848	3	1	7	0	2	0
19		635 909	229 563 149	3	1	5	3	1	0
23		390 876	206 773 404	3	1	7	1	1	0
$2 \cdot 3$	$d_K \equiv 3\,(9)$	5 963 493	214 685 748	3	1	7	2	0	0
$2 \cdot 3$	$d_K \equiv 6\,(9)$	4 305 957	155 014 452	0	4	6	3	0	0
$2 \cdot 5$		363 397	36 339 700	3	1	6	3	0	0
$2 \cdot 7$		358 285	70 223 860	4	0	7	2	0	0
$3 \cdot 5$	$d_K \equiv 3\,(9)$	4 845 432	1 090 222 200	3	1	6	3	0	0
$3 \cdot 5$	$d_K \equiv 6\,(9)$	1 646 817	370 533 825	3	1	6	3	0	0
$2 \cdot 3^2$	$d_K \equiv 1\,(3)$	2 142 445	694 152 180	3	1	6	3	0	0
$2 \cdot 3^2$	$d_K \equiv 2\,(3)$	635 909	206 034 516	3	1	6	3	0	0
$2 \cdot 3^2$	$d_K \equiv 6\,(9)$	2 538 285	822 404 340	3	1	6	3	0	0
$3 \cdot 7$	$d_K \equiv 3\,(9)$	3 597 960	1 586 700 360	3	1	6	3	0	0
$3 \cdot 7$	$d_K \equiv 6\,(9)$	3 122 232	1 376 904 312	0	4	6	3	0	0
$2 \cdot 11$		2 706 373	1 309 884 532	3	1	6	3	0	0

The result of the investigations is summarized in Table 5, which clearly shows that $d_L =$ **18 251 060**, for $d_K = 4\,562\,765$ and the smallest possible conductor $f = 2$ bigger than the conductor $f = 1$ of unramified extensions N/K, is the desired **minimal discriminant** of a totally real cubic field with ramified extension N/K, DPF type α_1 and $U_N = U_0$. The information has been computed with class field theoretic routines of Magma [8].

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