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Differential Principal Factors of Number Field Extensions

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Towers of p-Class Fields over Algebraic Number Fields

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1. Introduction

The intention of this lecture is to establish a common theoretical framework for the classification of

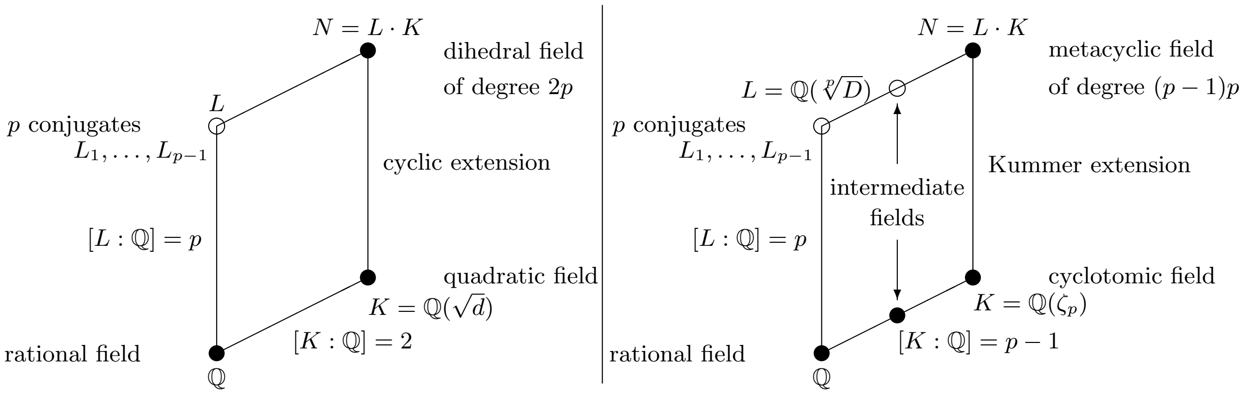
- dihedral fields $N/\mathbb{Q}$ of degree $2p$ with an odd prime p, viewed as p-ring class fields over a quadratic field K, and
- pure metacyclic fields $N = K(\sqrt[p]{D})$ of degree $(p-1)\cdot p$ with an odd prime p, viewed as Kummer extensions of a cyclotomic field $K = \mathbb{Q}(\zeta_p)$,

by the following arithmetical invariants:

(1) the $\mathbb{F}_p$-dimensions of subspaces of the space $\mathcal{P}_{N/K}/\mathcal{P}_K$ of primitive ambiguous principal ideals, which are also called *differential principal factors*, of N/K,

(2) the *capitulation kernel* $\ker(T_{N/K})$ of the transfer homomorphism $T_{N/K} : \mathrm{Cl}_p(K) \to \mathrm{Cl}_p(N)$ of p-classes, and

(3) the *Galois cohomology* $\mathrm{H}^0(G, U_N)$, $\mathrm{H}^1(G, U_N)$ of the unit group U_N as a module over the automorphism group $G = \mathrm{Gal}(N/K) \simeq C_p$.



2. Primitive ambiguous ideals

Let $p \geq 2$ be a prime number, and N/K be a relative extension of number fields with degree p, *not* necessarily Galois.

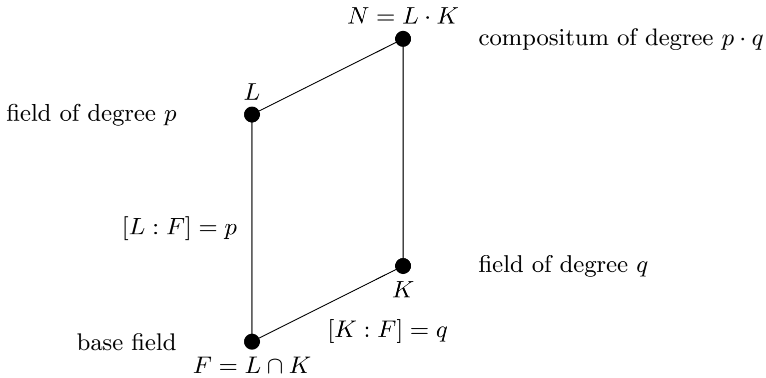
Definition 2.1. The group $\mathcal{I}_N$ of fractional ideals of N contains the *subgroup of ambiguous ideals* of N/K, $\mathcal{I}_{N/K} := \{\mathfrak{A} \in \mathcal{I}_N \mid \mathfrak{A}^p \in \mathcal{I}_K\}$. The quotient $\mathcal{I}_{N/K}/\mathcal{I}_K$ is called the $\mathbb{F}_p$-*vector space of primitive ambiguous ideals* of N/K.

Proposition 2.1. *Let $\mathfrak{L}_1, \ldots, \mathfrak{L}_t$ be the totally ramified prime ideals of N/K, then a basis and the dimension of $\mathcal{I}_{N/K}/\mathcal{I}_K$ over $\mathbb{F}_p$ are finite and given by*

$$\mathcal{I}_{N/K}/\mathcal{I}_K \simeq \prod_{i=1}^{t} \left(\langle\mathfrak{L}_i\rangle/\langle\mathfrak{L}_i^p\rangle\right) \simeq \mathbb{F}_p^t, \quad \dim_{\mathbb{F}_p}(\mathcal{I}_{N/K}/\mathcal{I}_K) = t,$$

whereas $\mathcal{I}_{N/K}$ is an infinite abelian group containing $\mathcal{I}_K$.

Proof. By the definition of $\mathcal{I}_{N/K}$, the quotient $\mathcal{I}_{N/K}/\mathcal{I}_K$ is an *elementary* abelian p-group. By the decomposition law for prime ideals of K in N, the space $\mathcal{I}_{N/K}/\mathcal{I}_K$ is generated by the *totally ramified* prime ideals (with ramification index $e = p$) of N/K: $\mathcal{I}_{N/K} = \langle\mathfrak{L} \in \mathbb{P}_N \mid \mathfrak{L}^p \in \mathbb{P}_K\rangle \cdot \mathcal{I}_K$. According to the theorem on prime ideals dividing the discriminant, the number t is *finite*. □



If L is another subfield of N such that $N = L \cdot K$ is the compositum of L and K, and N/L is of degree q *coprime* to p, then the relative norm homomorphism $N_{N/L}$ induces an *epimorphism*

$$N_{N/L} : \mathcal{I}_{N/K}/\mathcal{I}_K \to \mathcal{I}_{L/F}/\mathcal{I}_F, \tag{2.1}$$

where $F := L \cap K$ denotes the intersection of L and K. According to the isomorphism theorem, we have proved:

Theorem 2.1. *There are two isomorphisms between $\mathbb{F}_p$-vector spaces, quotient and direct product:*

$$\begin{aligned}(\mathcal{I}_{N/K}/\mathcal{I}_K)/\ker(N_{N/L}) &\simeq \mathcal{I}_{L/F}/\mathcal{I}_F,\\ \mathcal{I}_{N/K}/\mathcal{I}_K &\simeq (\mathcal{I}_{L/F}/\mathcal{I}_F) \times \ker(N_{N/L}).\end{aligned} \tag{2.2}$$

Definition 2.2. Since the relative different of N/K is essentially given by $\mathfrak{D}_{N/K} = \prod_{i=1}^{t} \mathfrak{L}_i^{p-1}$ the space $\mathcal{I}_{N/K}/\mathcal{I}_K \simeq \prod_{i=1}^{t} (\langle \mathfrak{L}_i \rangle / \langle \mathfrak{L}_i^p \rangle)$ of primitive ambiguous ideals of N/K is also called the space of *differential factors* of N/K. The two subspaces in the direct product decomposition of $\mathcal{I}_{N/K}/\mathcal{I}_K$ in formula (2.2) are called
subspace $\mathcal{I}_{L/F}/\mathcal{I}_F$ of *absolute* differential factors of L/F and subspace $\ker(N_{N/L})$ of *relative* differential factors of N/K.

2.1. **Splitting off the norm kernel.** The second isomorphism in formula (2.2) causes a *dichotomic decomposition* of the space $\mathcal{I}_{N/K}/\mathcal{I}_K$ into two components, whose dimensions can be given under the following conditions:

Theorem 2.2. *Let p be an **odd** prime and put $q = 2$. Among the prime ideals of L which are totally ramified over F, denote by $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ those which split in N, $\mathfrak{p}_i\mathcal{O}_N = \mathfrak{P}_i\mathfrak{P}_i'$ for $1 \le i \le s$, and by $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ those which remain inert in N, $\mathfrak{q}_j\mathcal{O}_N = \mathfrak{Q}_j$ for $1 \le j \le n$. Then the space $\mathcal{I}_{N/K}/\mathcal{I}_K$ is the direct product of the subspace $\mathcal{I}_{L/F}/\mathcal{I}_F$ of **absolute differential factors** of L/F and the subspace $\ker(N_{N/L})$ of **relative differential factors** of N/K, whose bases over $\mathbb{F}_p$ can be given by*

$$\mathcal{I}_{L/F}/\mathcal{I}_F \simeq \prod_{i=1}^{s} (\langle\mathfrak{p}_i\rangle/\langle\mathfrak{p}_i^p\rangle) \times \prod_{j=1}^{n} (\langle\mathfrak{q}_j\rangle/\langle\mathfrak{q}_j^p\rangle) \simeq \mathbb{F}_p^{s+n},$$

$$\ker(N_{N/L}) \simeq \prod_{i=1}^{s} \Big(\langle\mathfrak{P}_i(\mathfrak{P}_i')^{p-1}\rangle/\langle(\mathfrak{P}_i(\mathfrak{P}_i')^{p-1})^p\rangle\Big) \simeq \mathbb{F}_p^s.$$

Consequently, the complete space of differential factors has the dimension $\dim_{\mathbb{F}_p}(\mathcal{I}_{N/K}/\mathcal{I}_K) = n + 2s$.

Proof. Whereas the qualitative formula (2.2) is valid for any prime $p \ge 2$ and any integer $q > 1$ with $\gcd(p, q) = 1$, the quantitative description of the norm kernel $\ker(N_{N/L})$ is only feasible for $q = 2$ and an odd prime $p \ge 3$. Replacing N by L and K by F in formula (2.2), we get $t = n + s$ and thus the first isomorphism. For N and K, however, we obtain $t = n + 2s$. If $s = 0$ (none of the totally ramified primes of L/F splits in N), then the induced norm mapping $N_{N/L}$ in formula (2.1) is an isomorphism. $\square$

3. Primitive ambiguous principal ideals

The preceding result concerned *primitive ambiguous* **ideals** of N/K, which can be interpreted as ideal factors of the *relative different* $\mathfrak{D}_{N/K}$. Formula (2.1) and Theorem 2.1 show that the $\mathbb{F}_p$-dimension of the space $\mathcal{I}_{N/K}/\mathcal{I}_K$ increases indefinitely with the number t of totally ramified primes of N/K.

Now we restrict our attention to the space $\mathcal{P}_{N/K}/\mathcal{P}_K$ of *primitive ambiguous* **principal ideals** or **differential principal factors** (DPF) of N/K. We shall see that fundamental constraints from Galois cohomology prohibit an infinite growth of its dimension over $\mathbb{F}_p$, for quadratic fields K.

3.1. **Splitting off the capitulation kernel.** We have to cope with a difficulty which arises in the case of a non-trivial class group $\mathrm{Cl}(K) = \mathcal{I}_K/\mathcal{P}_K > 1$, because then $\mathcal{P}_{N/K}/\mathcal{P}_K$ cannot be viewed as a subspace of $\mathcal{I}_{N/K}/\mathcal{I}_K$. Therefore we must separate the *capitulation kernel* of N/K, that is the kernel of the *transfer* homomorphism

$$T_{N/K} : \mathrm{Cl}(K) \to \mathrm{Cl}(N),\ \mathfrak{a} \cdot \mathcal{P}_K \mapsto (\mathfrak{a}\mathcal{O}_N) \cdot \mathcal{P}_N,$$

which extends classes of K to classes of N:

$$\ker(T_{N/K}) = \{\mathfrak{a}\cdot\mathcal{P}_K \mid (\exists\, A \in N)\, \mathfrak{a}\mathcal{O}_N = A\mathcal{O}_N\} = (\mathcal{I}_K\cap\mathcal{P}_N)/\mathcal{P}_K.$$

On the one hand, $\ker(T_{N/K}) = (\mathcal{I}_K\cap\mathcal{P}_N)/\mathcal{P}_K$ is a subgroup of $\mathcal{I}_K/\mathcal{P}_K = \mathrm{Cl}(K)$, consisting of capitulating ideal classes of K. On the other hand, since $\mathcal{I}_K \le \mathcal{I}_{N/K}$ consists of ambiguous ideals of N/K, $\ker(T_{N/K}) = (\mathcal{I}_K \cap \mathcal{P}_N)/\mathcal{P}_K$ is a subgroup of $\mathcal{P}_{N/K}/\mathcal{P}_K$, consisting of special primitive ambiguous principal ideals of N/K, and we can form the quotient

$$\begin{aligned}(\mathcal{P}_{N/K}/\mathcal{P}_K)/\big((\mathcal{I}_K \cap \mathcal{P}_N)/\mathcal{P}_K\big) &\simeq \mathcal{P}_{N/K}/(\mathcal{I}_K \cap \mathcal{P}_N)\\ &= \mathcal{P}_{N/K}/(\mathcal{I}_K \cap \mathcal{P}_{N/K}) \simeq (\mathcal{P}_{N/K} \cdot \mathcal{I}_K)/\mathcal{I}_K.\end{aligned}$$

This quotient relation of $\mathbb{F}_p$-vector spaces is equivalent to a direct product relation

$$\mathcal{P}_{N/K}/\mathcal{P}_K \simeq (\mathcal{P}_{N/K} \cdot \mathcal{I}_K)/\mathcal{I}_K \times \ker(T_{N/K}). \tag{3.1}$$

Since $(\mathcal{P}_{N/K} \cdot \mathcal{I}_K)/\mathcal{I}_K \le \mathcal{I}_{N/K}/\mathcal{I}_K$ is an actual inclusion, the factorization of $\mathcal{I}_{N/K}/\mathcal{I}_K$ in formula (2.2) restricts to a factorization

$$(\mathcal{P}_{N/K}\cdot\mathcal{I}_K)/\mathcal{I}_K \simeq (\mathcal{P}_{L/F}/\mathcal{P}_F)\times\Big(\ker(N_{N/L})\cap\big((\mathcal{P}_{N/K}\cdot\mathcal{I}_K)/\mathcal{I}_K\big)\Big), \tag{3.2}$$

provided that F is a field with trivial class group $\mathrm{Cl}(F)$, that is $\mathcal{I}_F = \mathcal{P}_F$. Combining the formulas (3.1) and (3.2) for the rational base field $F = \mathbb{Q}$, we obtain:

Theorem 3.1. *There is a* ***trichotomic decomposition*** *of the space $\mathcal{P}_{N/K}/\mathcal{P}_K$ of differential principal factors of N/K into three components,*

$$\mathcal{P}_{N/K}/\mathcal{P}_K \simeq \mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}\times\Big(\ker(N_{N/L})\cap\big((\mathcal{P}_{N/K}\mathcal{I}_K)/\mathcal{I}_K\big)\Big)\times\ker(T_{N/K}), \tag{3.3}$$

the ***absolute principal factors****, $\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}$, of $L/\mathbb{Q}$,*
the ***relative principal factors****, $\ker(N_{N/L})\cap\big((\mathcal{P}_{N/K}\mathcal{I}_K)/\mathcal{I}_K\big)$, of N/K, and*
the ***capitulation kernel****, $\ker(T_{N/K})$, of N/K.*

3.2. **Galois cohomology.** For establishing a quantitative version of the qualitative formula (3.3), we suppose that N/K is a cyclic relative extension of *odd* prime degree p and we use the Galois cohomology of the unit group U_N as a module over the automorphism group $G = \mathrm{Gal}(N/K) = \langle\sigma\rangle \simeq C_p$. In fact, we combine a theorem of Iwasawa [8] on the first cohomology $\mathrm{H}^1(G, U_N)$ with a theorem of Hasse [3] on the Herbrand quotient of U_N [6], and we use Dirichlet's theorem on the torsion-free unit rank of K:

$$\mathrm{H}^1(G, U_N) \simeq (U_N \cap \ker(N_{N/K}))/U_N^{\sigma-1} \simeq \mathcal{P}_{N/K}/\mathcal{P}_K \text{ (Iwasawa)},$$

$$\#\mathrm{H}^0(G, U_N) = (U_K : N_{N/K}(U_N)) = p^U, \ 0 \le U \le r_1 + r_2 - \theta,$$

$$\frac{\#\mathrm{H}^1(G, U_N)}{\#\mathrm{H}^0(G, U_N)} = [N : K] = p \quad \text{(Hasse)},$$

where (r_1, r_2) is the signature of K, and $\theta = 0$ if K contains the pth roots of unity, but $\theta = 1$ else.

Corollary 3.1. *If N/K is cyclic of odd prime degree $p \ge 3$, then the $\mathbb{F}_p$-dimensions of the spaces of differential principal factors in Theorem 3.1 are connected by the* ***fundamental equation***

$$U + 1 = A + R + C, \quad \textit{where} \tag{3.4}$$

$A := \dim_{\mathbb{F}_p}(\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}})$,

$R := \dim_{\mathbb{F}_p}\Big(\ker(N_{N/L}) \cap \big((\mathcal{P}_{N/K}\mathcal{I}_K)/\mathcal{I}_K\big)\Big)$, *and*

$C := \dim_{\mathbb{F}_p}(\ker(T_{N/K}))$.

Corollary 3.2. *Under the assumptions $p \geq 3$, $q = 2$ of Theorem 2.1, in particular for N dihedral of degree $2p$, the dimensions in Corollary 3.1 are bounded by the following* ***estimates***

(3.5)

$$0 \leq A \leq \min(n+s, m),\ 0 \leq R \leq \min(s, m),\ 0 \leq C \leq \min(\varrho_p, m),$$

where $m := 1 + r_1 + r_2 - \theta$ is the cohomological maximum of $U + 1$, and $\varrho_p := \mathrm{rank}_p(\mathrm{Cl}(K))$. In particular,
$m = 2$ for real quadratic K with $(r_1, r_2) = (2, 0)$, $\theta = 1$,
$m = 1$ for imaginary quadratic K ($\neq \mathbb{Q}(\sqrt{-3})$ if $p = 3$) with $(r_1, r_2) = (0, 1)$, $\theta = 1$.

Remark 3.1. For N pure metacyclic of degree $(p-1)p$, the space $\mathcal{P}_{L/\mathbb{Q}}/\mathcal{P}_{\mathbb{Q}}$ of absolute principal factors contains the one-dimensional subspace $\Delta = \langle \sqrt[p]{D} \rangle$ generated by the radicals, and thus

(3.6)

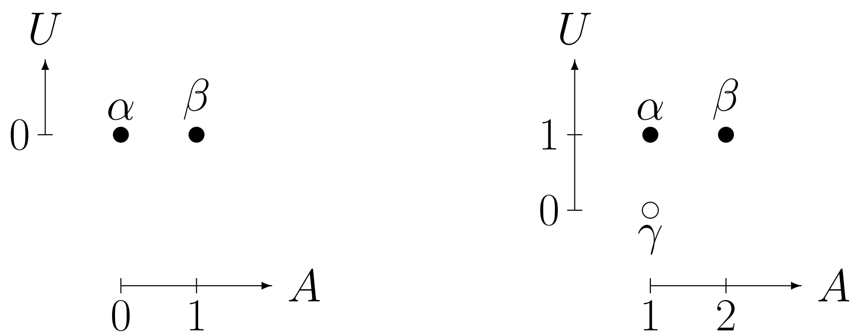
$$1 \leq A \leq \min(t, m),\ 0 \leq R \leq m-1,\ 0 \leq C \leq \min(\varrho_p, m-1),$$

where $m = \frac{p+1}{2}$ for cyclotomic K with $(r_1, r_2) = (0, \frac{p-1}{2})$.
In particular $C = 0$ for a regular prime p, for instance $p < 37$.

Remark 3.2. We mentioned that in general $\mathcal{P}_{N/K}/\mathcal{P}_K$ cannot be viewed as a subspace of $\mathcal{I}_{N/K}/\mathcal{I}_K$. In fact, for a dihedral field N which is unramified with conductor $f = 1$ over K, we have $n = s = 0$, consequently $A = R = 0$, and $\mathcal{I}_{N/K}/\mathcal{I}_K = 0$ is the nullspace, whereas $\mathcal{P}_{N/K}/\mathcal{P}_K = \ker(T_{N/K})$ is at least one-dimensional, according to Hilbert's Theorem 94 [7], and at most two-dimensional, by the estimate $C \leq \min(\varrho_p, m) \leq \min(\varrho_p, 2) \leq 2$.

3.3. **Differential principal factorization (DPF) types of complex dihedral fields.** Let p be an odd prime. We recall the classification theorem for *pure cubic* fields $L = \mathbb{Q}(\sqrt[3]{D})$ and their Galois closure $N = \mathbb{Q}(\zeta_3, \sqrt[3]{D})$, that is the metacyclic case $p = 3$. The *coarse* classification of N according to the cohomological invariants U and A alone is closely related to the classification of *simply real dihedral* fields of degree $2p$ with any odd prime p by Nicole Moser [19, Dfn. III.1 and Prop. III.3, p. 61], as illustrated in Figure 1. The coarse types α and β are completely analogous in both cases. The additional type γ is required for pure cubic fields, because there arises the possibility that the primitive cube root of unity ζ_3 occurs as relative norm $N_{N/K}(Z)$ of a unit $Z \in U_N$. Due to the existence of radicals in the pure cubic case, the $\mathbb{F}_p$-dimension A of the vector space of absolute DPF exceeds the corresponding dimension for simply real dihedral fields by one.

FIGURE 1. Classification of Simply Real Dihedral and Pure Cubic Fields



The *fine* classification of N according to the invariants U, A, R and C in the simply real dihedral situation with $U+1 = A+R+C$ splits type α with $A = 0$ further in type α_1 with $C = 1$ (capitulation) and type α_2 with $R = 1$ (relative DPF). In the pure cubic situation, however, no further splitting occurs, since $C = 0$, and $R = U + 1 - A$ is determined uniquely by U and A already. We oppose the two classifications in the following theorems.

Theorem 3.2. *Each simply real dihedral field $N/\mathbb{Q}$ of absolute degree $[N : \mathbb{Q}] = 2p$ with an odd prime p belongs to precisely one of the following 3 differential principal factorization types, in dependence on the triplet (A, R, C):*

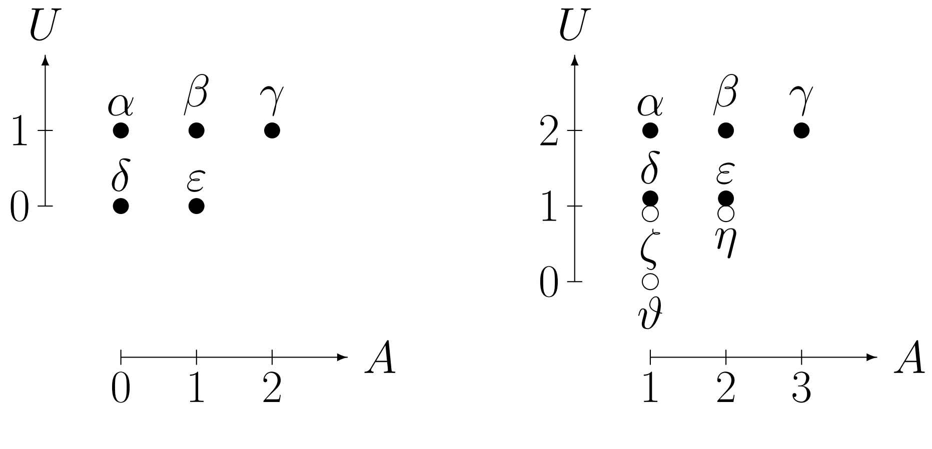
Type	U	$U + 1 = A + R + C$	A	R	C
α_1	0	1	0	0	1
α_2	0	1	0	1	0
β	0	1	1	0	0

Theorem 3.3. *Each pure metacyclic field $N = \mathbb{Q}(\zeta_3, \sqrt[3]{D})$ of absolute degree $[N : \mathbb{Q}] = 6$ with cube free radicand $D \in \mathbb{Z}$, $D \geq 2$, belongs to precisely one of the following 3 differential principal factorization types, in dependence on the invariant U and the pair (A, R):*

Type	U	$U + 1 = A + R$	A	R
α	1	2	1	1
β	1	2	2	0
γ	0	1	1	0

3.4. **Differential principal factorization (DPF) types of real dihedral fields.** Now we state the classification theorem for *pure quintic* fields $L = \mathbb{Q}(\sqrt[5]{D})$ and their Galois closure $N = \mathbb{Q}(\zeta_5, \sqrt[5]{D})$, that is the metacyclic case $p = 5$. The *coarse* classification of N according to the invariants U and A alone is closely related to the classification of *totally real dihedral* fields of degree $2p$ with any odd prime p by Nicole Moser [19, Thm. III.5, p. 62], as illustrated in Figure 2. The coarse types α, β, γ, δ, ε are completely analogous in both cases. Additional types ζ, η, ϑ are required for pure quintic fields, because there arises the possibility that the primitive fifth root of unity ζ_5 occurs as relative norm $N_{N/K}(Z)$ of a unit $Z \in U_N$. Due to the existence of radicals in the pure quintic case, the $\mathbb{F}_p$-dimension A of the vector space of absolute DPF exceeds the corresponding dimension for totally real dihedral fields by one (see Remark 3.1).

FIGURE 2. Classification of Totally Real Dihedral and Pure Quintic Fields



The *fine* classification of N according to the invariants U, A, R and C in the totally real dihedral situation with $U+1 = A+R+C$ splits type α with $U=1$, $A=0$ further in type α_1 with $C=2$ (double capitulation), type α_2 with $C=R=1$ (mixed capitulation and relative DPF), type α_3 with $R=2$ (double relative DPF), type β with $U=A=1$ in type β_1 with $C=1$ (capitulation), type β_2 with $R=1$ (relative DPF), and type δ with $U=A=0$ in type δ_1 with $C=1$ (capitulation), type δ_2 with $R=1$ (relative DPF).

Theorem 3.4. *Each totally real dihedral field $N/\mathbb{Q}$ of absolute degree $[N:\mathbb{Q}]=2p$ with an odd prime p belongs to precisely one of the following* 9 *differential principal factorization types, in dependence on the invariant U and the triplet (A,R,C).*

Type	U	$U+1=A+R+C$	A	R	C
α_1	1	2	0	0	2
α_2	1	2	0	1	1
α_3	1	2	0	2	0
β_1	1	2	1	0	1
β_2	1	2	1	1	0
γ	1	2	2	0	0
δ_1	0	1	0	0	1
δ_2	0	1	0	1	0
ε	0	1	1	0	0

Proof. Consequence of the Corollaries 3.1 and 3.2. See also [19, Thm. III.5, p. 62] and [11]. □

In the pure quintic situation with $U+1=A+I+R$ [17], however, we arrive at the following theorem.

Theorem 3.5. *Each pure metacyclic field $N=\mathbb{Q}(\zeta_5,\sqrt[5]{D})$ of absolute degree $[N:\mathbb{Q}]=20$ with 5-th power free radicand $D\in\mathbb{Z}$, $D\geq 2$, belongs to precisely one of the following 13 differential principal factorization types, in dependence on the invariant U and the triplet (A,I,R).*

Type	U	$U+1=A+I+R$	A	I	R
α_1	2	3	1	0	2
α_2	2	3	1	1	1
α_3	2	3	1	2	0
β_1	2	3	2	0	1
β_2	2	3	2	1	0
γ	2	3	3	0	0
δ_1	1	2	1	0	1
δ_2	1	2	1	1	0
ε	1	2	2	0	0
ζ_1	1	2	1	0	1
ζ_2	1	2	1	1	0
η	1	2	2	0	0
ϑ	0	1	1	0	0

The types δ_1, δ_2, ε are characterized additionally by $\zeta_5\notin N_{N/K}(U_N)$, and the types ζ_1, ζ_2, η by $\zeta_5\in N_{N/K}(U_N)$.

Proof. The proof is given in [17, Thm. 6.1]. □

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