COMPLEX QUADRATIC FIELDS OF TYPE $(3, 3, 3)$

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ABSTRACT. Due to Koch and Venkov, it is known that a complex quadratic field with 3-class rank 3 has an infinite 3-class field tower. Diaz y Diaz and Buell have determined the smallest absolute discriminants of such fields. Below the bound 10^7 there exist 25 discriminants of this kind, and 14 of the corresponding fields have a 3-class group of elementary abelian type $(3, 3, 3)$. For each of these 14 fields, we determine the type of 3-principalization in unramified cyclic cubic extensions, the structure of the 3-class groups of these extensions, and the metabelian Galois group G of the second Hilbert 3-class field. We provide evidence for a wealth of structure in the set of infinite 3-class field towers by showing that the 14 groups G are pairwise non-isomorphic.

1. DISCRIMINANTS $-10^7 < d < 0$ OF FIELDS $K = \mathbb{Q}(\sqrt{d})$ WITH RANK $r_3(K) = 3$

Since our aim is to investigate tendencies for the coclass of second and higher p-class groups $\mathrm{Gal}(\mathrm{F}_p^n(K)|K)$, $n \geq 2$, [14, 16] of a series of algebraic number fields K with infinite p-class field tower, for an odd prime $p \geq 3$, the most obvious choice which suggests itself is to take the smallest possible prime $p = 3$ and to select complex quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, having the simplest possible 3-class group of rank three, that is, of elementary abelian type $(3, 3, 3)$.

The reason is that Koch and Venkov [11] have improved the lower bound of Golod, Shafarevich [18, 9] and Vinberg [20] for the p-class rank, which ensures an infinite p-class tower of a complex quadratic field, from four to three.

However, quadratic fields with 3-rank three are sparse. Diaz y Diaz and Buell [5, 19, 4] have determined the minimal absolute discriminant of such fields to be 3 321 607.

To provide an independent verification, we use the Magma computer algebra system [2, 3, 12] for compiling a list of all quadratic fundamental discriminants $-10^7 < d < 0$ of fields $K = \mathbb{Q}(\sqrt{d})$ with 3-class rank $r_3(K) = 3$. In 16 hours of CPU time we obtain the 25 desired discriminants and the abelian type invariants of the corresponding 3-class groups $\mathrm{Cl}_3(K)$, and also of the complete class groups $\mathrm{Cl}(K)$, as given in Table 1.

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There are 14 discriminants, starting with $d = -4\,447\,704$, such that $\mathrm{Cl}_3(K)$ is elementary abelian of type $(3, 3, 3)$, and 10 discriminants, starting with $-3\,321\,607$, such that $\mathrm{Cl}_3(K)$ is of non-elementary type $(9, 3, 3)$. For the single discriminant $d = -5\,153\,431$, we have a 3-class group of type $(27, 3, 3)$. We have published this information in the Online Encyclopedia of Integer Sequences (OEIS), A244574 and A244575.

TABLE 1. Data collection

No.	discriminant d	$\mathrm{Cl}_3(K)$	$\mathrm{Cl}(K)$
1	$-3\,321\,607$	$(9, 3, 3)$	$(63, 3, 3)$
2	$-3\,640\,387$	$(9, 3, 3)$	$(18, 3, 3)$
3	$-4\,019\,207$	$(9, 3, 3)$	$(207, 3, 3)$
4	$-4\,447\,704$	$(3, 3, 3)$	$(24, 6, 6)$
5	$-4\,472\,360$	$(3, 3, 3)$	$(30, 6, 6)$
6	$-4\,818\,916$	$(3, 3, 3)$	$(48, 3, 3)$
7	$-4\,897\,363$	$(3, 3, 3)$	$(33, 3, 3)$
8	$-5\,048\,347$	$(9, 3, 3)$	$(18, 6, 3)$
9	$-5\,067\,967$	$(3, 3, 3)$	$(69, 3, 3)$
10	$-5\,153\,431$	$(27, 3, 3)$	$(216, 3, 3)$
11	$-5\,288\,968$	$(9, 3, 3)$	$(72, 3, 3)$
12	$-5\,769\,988$	$(3, 3, 3)$	$(12, 6, 6)$
13	$-6\,562\,327$	$(9, 3, 3)$	$(126, 3, 3)$
14	$-7\,016\,747$	$(9, 3, 3)$	$(99, 3, 3)$
15	$-7\,060\,148$	$(3, 3, 3)$	$(60, 6, 3)$
16	$-7\,503\,391$	$(9, 3, 3)$	$(90, 6, 3)$
17	$-7\,546\,164$	$(9, 3, 3)$	$(18, 6, 6, 2)$
18	$-8\,124\,503$	$(9, 3, 3)$	$(261, 3, 3)$
19	$-8\,180\,671$	$(3, 3, 3)$	$(159, 3, 3)$
20	$-8\,721\,735$	$(3, 3, 3)$	$(60, 6, 6)$
21	$-8\,819\,519$	$(3, 3, 3)$	$(276, 3, 3)$
22	$-8\,992\,363$	$(3, 3, 3)$	$(48, 3, 3)$
23	$-9\,379\,703$	$(3, 3, 3)$	$(210, 3, 3)$
24	$-9\,487\,991$	$(3, 3, 3)$	$(381, 3, 3)$
25	$-9\,778\,603$	$(3, 3, 3)$	$(48, 3, 3)$

2. Arithmetic invariants of fields $K=\mathbb{Q}(\sqrt{d})$ with $\mathrm{Cl}_3(K)\simeq(3,3,3)$

After the preliminary data collection in section 1, we restrict ourselves to the 14 cases with elementary abelian 3-class group of type $(3,3,3)$. The complex quadratic field $K=\mathbb{Q}(\sqrt{d})$ possesses 13 unramified cyclic cubic extensions $L_1,\ldots,L_{13}$ with dihedral absolute Galois group $\mathrm{Gal}(L_i|\mathbb{Q})$ of order six. Based on Fieker's technique [6], we use the Magma computer algebra system [3, 12] to construct these extensions and to calculate their arithmetical invariants. In Table 2, which is continued in Table 3 on the following page, we present the kernel $\varkappa_i$ of the 3-principalization of K in L_i [13, 14], the occupation numbers $o(\varkappa)_i$ of the principalization kernels [15], and the abelian type invariants τ_i, resp. τ_i^0, of the 3-class group $\mathrm{Cl}_3(L_i)$, resp. $\mathrm{Cl}_3(K_i)$, for each $1\le i\le 13$ [16]. Here, we denote by K_i the unique real non-Galois absolutely cubic subfield of L_i. For brevity, we give 3-logarithms of abelian type invariants and we denote iteration by formal exponents, for example, $21^4:=(9,3,3,3,3)$ or $2^21^2:=(9,9,3,3)$.

Table 2. Pattern recognition

No.	discriminant												
i	1	2	3	4	5	6	7	8	9	10	11	12	13
1	$d=-4\,447\,704$												
$\varkappa$	8	1	8	8	10	8	6	13	8	2	10	8	9
$o(\varkappa)$	1	1	0	0	0	1	0	6	1	2	0	0	1
τ	2^21^2	21^4	2^21^2	32^21	2^21^2	21^4	2^21^2	2^21^2	21^4	21^4	21^4	2^21^2	2^21^2
τ^0	1^2	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2
2	$d=-4\,472\,360$												
$\varkappa$	1	12	6	13	6	10	4	13	10	10	1	8	4
$o(\varkappa)$	2	0	0	2	0	2	0	1	0	3	0	1	2
τ	2^21^2	2^21^2	2^21^2	21^4	2^21^2	21^4	21^4	2^21^2	2^21^2	2^21^2	32^21	2^21^2	21^4
τ^0	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	21	1^2	1^2
3	$d=-4\,818\,916$												
$\varkappa$	6	9	13	1	5	6	9	4	11	7	1	3	4
$o(\varkappa)$	2	0	1	2	1	2	1	0	2	0	1	0	1
τ	2^21^2	2^21^2	2^21^2	21^4	21^4	2^21^2	2^21^2	2^21^2	431^3	2^21^2	21^4	32^21	2^21^2
τ^0	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	31	1^2	1^2	21	1^2
4	$d=-4\,897\,363$												
$\varkappa$	3	8	11	2	6	6	12	7	2	2	9	13	6
$o(\varkappa)$	0	3	1	0	0	3	1	1	1	0	1	1	1
τ	2^21^2	321^3	431^3	2^21^2	21^4	2^21^2	32^21	21^4	2^21^2	2^21^2	2^21^2	2^21^2	2^21^2
τ^0	1^2	21	31	1^2	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2
5	$d=-5\,067\,967$												
$\varkappa$	8	6	9	2	3	7	12	7	1	4	3	9	4
$o(\varkappa)$	1	1	2	2	0	1	2	1	2	0	0	1	0
τ	21^4	21^4	2^21^2	21^4	2^21^2	2^21^2	2^21^2	32^21	2^21^2	21^4	21^4	2^21^2	2^21^2
τ^0	1^2	1^2	1^2	1^2	1^2	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2
6	$d=-5\,769\,988$												
$\varkappa$	12	11	7	6	1	1	10	10	9	6	4	3	13
$o(\varkappa)$	2	0	1	1	0	2	1	0	1	2	1	1	1
τ	321^3	2^21^2	21^4	321^3	2^21^2	21^4	21^4	2^21^2	2^21^2	2^21^2	2^21^2	21^4	32^21
τ^0	21	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	21

Table 3. Pattern recognition (continued)

No.	discriminant												
i	1	2	3	4	5	6	7	8	9	10	11	12	13
7	$d = -7\,060\,148$												
$\varkappa$	2	4	4	9	4	10	8	10	10	1	6	8	3
$o(\varkappa)$	1	1	1	3	0	1	0	2	1	3	0	0	0
τ	21^4	21^4	32^21	2^21^2	32^21	21^4	2^21^2	321^3	21^4	21^4	2^21^2	2^21^2	321^3
τ^0	1^2	1^2	21	1^2	21	1^2	1^2	21	1^2	1^2	1^2	1^2	21
8	$d = -8\,180\,671$												
$\varkappa$	12	9	2	6	10	6	8	2	10	10	9	11	4
$o(\varkappa)$	0	2	0	1	0	2	0	1	2	3	1	1	0
τ	321^3	2^21^2	21^4	2^21^2	2^21^2	2^21^2	2^21^2	21^4	2^21^2	2^21^2	2^21^2	21^4	2^21^2
τ^0	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2
9	$d = -8\,721\,735$												
$\varkappa$	5	2	5	1	10	13	4	7	11	3	9	8	8
$o(\varkappa)$	1	1	1	1	2	0	1	2	1	1	1	0	1
τ	2^21^2	21^4	21^4	21^4	321^3	2^21^2	21^4	2^21^2	32^21	32^21	21^4	32^21	2^21^2
τ^0	1^2	1^2	1^2	1^2	21	1^2	1^2	1^2	21	21	1^2	21	1^2
10	$d = -8\,819\,519$												
$\varkappa$	2	7	8	12	4	12	9	5	5	3	10	6	10
$o(\varkappa)$	0	1	1	1	2	1	1	1	1	2	0	2	0
τ	2^21^2	21^4	32^21	2^21^2	2^21^2	2^21^2	2^21^2	2^21^2	21^4	2^21^2	2^21^2	2^21^2	1^6
τ^0	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2
11	$d = -8\,992\,363$												
$\varkappa$	12	10	2	12	9	5	10	10	2	12	6	9	7
$o(\varkappa)$	0	2	0	0	1	1	1	0	2	3	0	3	0
τ	2^21^2	21^4	2^21^2	2^21^2	32^21	2^21^2	21^4	21^4	21^4	2^21^2	2^21^2	21^4	2^21^2
τ^0	1^2	1^2	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2
12	$d = -9\,379\,703$												
$\varkappa$	8	11	8	13	9	5	6	1	2	13	4	12	3
$o(\varkappa)$	1	1	1	1	1	1	0	2	1	0	1	1	2
τ	21^4	2^21^2	321^3	2^21^2	21^4	21^4	2^21^2	21^4	21^4	2^21^2	2^21^2	2^21^2	2^21^2
τ^0	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2
13	$d = -9\,487\,991$												
$\varkappa$	4	2	2	11	13	9	12	9	8	1	1	12	3
$o(\varkappa)$	2	2	1	1	0	0	0	1	2	0	1	2	1
τ	2^21^2	2^21^2	2^21^2	2^21^2	321^3	2^21^2	2^21^2	21^4	2^21^2	21^4	2^21^2	2^21^2	2^21^2
τ^0	1^2	1^2	1^2	1^2	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2
14	$d = -9\,778\,603$												
$\varkappa$	10	6	6	9	9	10	8	10	13	5	12	6	10
$o(\varkappa)$	0	0	0	0	1	3	0	1	2	4	0	1	1
τ	2^21^2	321^3	21^4	32^21	32^21	2^21^2	2^21^2	2^21^2	21^4	21^4	2^21^2	2^21^2	2^21^2
τ^0	1^2	21	1^2	21	21	1^2	1^2	1^2	1^2	1^2	1^2	1^2	1^2

In Table 4, we classify each of the 14 complex quadratic fields $K = \mathbb{Q}(\sqrt{d})$ of type $(3,3,3)$ according to the occupation numbers of the abelian type invariants of the 3-class groups $\mathrm{Cl}_3(L_i)$ of the 13 unramified cyclic cubic extensions L_i. Whereas the dominant part of these groups is of order $3^6 = 729$, there always exist at least one and at most four distinguished groups of bigger order, usually $3^8 = 6561$ and occasionally even $3^{10} = 59049$, According to the number of distinguished groups, we speak about *uni-*, *bi-*, *tri-* or *tetra*-polarization. If the maximal value of the order is 3^8, then we have a *ground* state, otherwise an *excited* state.

TABLE 4. Cumulative form of abelian type invariants

No.	discriminant d	2^21^2	21^4	1^6	32^21	321^3	431^3	polarization	state
1	$-4\,447\,704$	7	5	0	1	0	0	uni	ground
2	$-4\,472\,360$	8	4	0	1	0	0	uni	ground
3	$-4\,818\,916$	8	3	0	1	0	1	bi	excited
4	$-4\,897\,363$	8	2	0	1	1	1	tri	excited
5	$-5\,067\,967$	7	5	0	1	0	0	uni	ground
6	$-5\,769\,988$	6	4	0	1	2	0	tri	ground
7	$-7\,060\,148$	4	5	0	2	2	0	tetra	ground
8	$-8\,180\,671$	9	3	0	0	1	0	uni	ground
9	$-8\,721\,735$	4	5	0	3	1	0	tetra	ground
10	$-8\,819\,519$	9	2	1	1	0	0	uni	ground
11	$-8\,992\,363$	7	5	0	1	0	0	uni	ground
12	$-9\,379\,703$	7	5	0	0	1	0	uni	ground
13	$-9\,487\,991$	10	2	0	0	1	0	uni	ground
14	$-9\,778\,603$	7	3	0	2	1	0	tri	ground

The information given in Table 4 consists of isomorphism invariants of the metabelian Galois group $G = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ of the second Hilbert 3-class field of K [14, 16]. Consequently, with respect to the 13 abelian type invariants of the 3-class groups $\mathrm{Cl}_3(L_i)$ alone, only the groups G for $d \in \{-4\,447\,704, -5\,067\,967, -8\,992\,363\}$ could be isomorphic. However, Tables 2 and 3 show that these three groups differ in another isomorphism invariant, the 3-principalization type $\varkappa$ [13, 15], since the corresponding maximal occupation numbers of the multiplet $o(\varkappa)$ are $6, 2, 3$, respectively. We summarize this result and its obvious conclusion in the following Theorem.

Theorem 2.1. *The* 14 *complex quadratic number fields* $K = \mathbb{Q}(\sqrt{d})$ *with* 3*-class groups* $\mathrm{Cl}_3(K)$ *of type* $(3,3,3)$ *and discriminants in the range* $-10^7 < d < 0$ *have pairwise non-isomorphic*

(1) *second and higher* 3*-class groups* $\mathrm{Gal}(\mathrm{F}_3^n(K)|K)$, $n \geq 2$,
(2) *infinite topological* 3*-class tower groups* $\mathrm{Gal}(\mathrm{F}_3^\infty(K)|K)$.

3. FINAL REMARK

We would like to emphasize that Theorem 2.1 provides evidence for a wealth of structure in the set of infinite 3-class field towers, which was unknown up to now, since the common practice is to consider a 3-class field tower as done when some criterion in the style of Golod-Shafarevich-Vinberg [18, 9, 20] or Koch-Venkov [11] ensures just its infinity. However, this perspective is very coarse and our result proves that it can be refined considerably.

References

[1] H. U. Besche, B. Eick, and E. A. O'Brien, *The SmallGroups Library — a Library of Groups of Small Order*, 2005, an accepted and refereed GAP 4 package, available also in MAGMA.

[2] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[3] W. Bosma, J. J. Cannon, C. Fieker, and A. Steels (eds.), *Handbook of Magma functions* (Edition 2.20, Sydney, 2014).

[4] D. A. Buell, *Class groups of quadratic fields*, Math. Comp. **30** (1976), no. 135, 610–623.

[5] F. Diaz y Diaz, *Sur les corps quadratiques imaginaires dont le* 3*-rang du groupe des classes est supérieur à* 1, Séminaire Delange-Pisot-Poitou, 1973/74, no. G15.

[6] C. Fieker, *Computing class fields via the Artin map*, Math. Comp. **70** (2001), no. 235, 1293–1303.

[7] The GAP Group, *GAP – Groups, Algorithms, and Programming — a System for Computational Discrete Algebra*, Version 4.4.12, Aachen, Braunschweig, Fort Collins, St. Andrews, 2008, (http://www.gap-system.org).

[8] G. Gamble, W. Nickel, and E. A. O'Brien, *ANU p-Quotient — p-Quotient and p-Group Generation Algorithms*, 2006, an accepted GAP 4 package, available also in MAGMA.

[9] E. S. Golod and I. R. Shafarevich, *On class field towers* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **28** (1964), no. 2, 261–272. (English transl. in Amer. Math. Soc. Transl. (2) **48** (1965), 91–102.)

[10] D. F. Holt, B. Eick, and E. A. O'Brien, *Handbook of computational group theory*, Discrete mathematics and its applications, Chapman and Hall/CRC, 2005.

[11] H. Koch und B. B. Venkov, *Über den p-Klassenkörperturm eines imaginär-quadratischen Zahlkörpers*, Astérisque **24–25** (1975), 57–67.

[12] The MAGMA Group, *MAGMA Computational Algebra System*, Version 2.20-7, Sydney, 2014, (http://magma.maths.usyd.edu.au).

[13] D. C. Mayer, Principalization in complex S_3-fields, *Congressus Numerantium* **80** (1991), 73–87 (Proceedings of the Twentieth Manitoba Conference on Numerical Mathematics and Computing, Winnipeg, Manitoba, Canada, 1990).

[14] D. C. Mayer, The second p-class group of a number field, *Int. J. Number Theory* **8** (2012), no. 2, 471–505, DOI 10.1142/S179304211250025X.

[15] D. C. Mayer, Transfers of metabelian p-groups, *Monatsh. Math.* **166** (2012), no. 3–4, 467–495, DOI 10.1007/s00605-010-0277-x.

[16] D. C. Mayer, *The distribution of second p-class groups on coclass graphs*, J. Théor. Nombres Bordeaux **25** (2013), no. 2, 401–456.

[17] E. A. O'Brien, *The p-group generation algorithm*, J. Symbolic Comput. **9** (1990), 677–698.

[18] I. R. Shafarevich, *Extensions with prescribed ramification points* (Russian), Publ. Math., Inst. Hautes Études Sci. **18** (1963), 71–95. (English transl. by J. W. S. Cassels in Amer. Math. Soc. Transl., II. Ser., **59** (1966), 128–149.)

[19] D. Shanks, *Class groups of the quadratic fields found by Diaz y Diaz*, Math. Comp. **30** (1976), 173–178.

[20] E. B. Vinberg, *On a theorem concerning the infinite-dimensionality of an associative algebra* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **29** (1965), 209–214. (English transl. in Amer. Math. Soc. Transl. (2) **82** (1969), 237–242.)

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