Der Wissenschaftsfonds.

Finite 3-Groups as viewed from Class Field Theory

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Date:	Friday, August 09, 2013
Time:	11:00 – 11:20, a. m.
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§ 0. Summary of Aims

(1) **Complete** determination of **all** finite 3-groups with transfer kernel type (TKT) E (and thus with abelianization of type $(3, 3)$)
(2) **New** parametrized power-commutator presentations of **all metabelian** 3-groups with TKT E (supplementing the presentations by Nebelung)
(3) **First** parametrized power-commutator presentations of non-metabelian 3-groups with TKT E, which are all of **derived length three**
(4) **First** explicit construction of covers and Schur covers of **all** metabelian 3-groups with TKT E
(5) **New** kind of periodicity of tree bifurcations, sporadic isolated Schur groups, and TKT-pruned coclass trees
(6) Construction of two **infinite pro-**3 **groups** which have all sporadic isolated Schur groups as finite quotients
(7) Evidence for an extensive class of complex quadratic fields having a 3-class field tower with **exactly three stages**

Acknowledgement

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§ 1. Classical Theorems

§ 1.0. Number Theory Notation

K an algebraic number field,
$p \geq 3$ an odd prime,
$\mathrm{Cl}_p(K)$ the p-class group of K,
$\mathrm{r}_p(K)$ the p-class rank of K.

$\mathrm{F}_p^m(K)$ the mth *Hilbert p-class field* of K, $m \geq 1$, that is, the maximal unramified p-extension of K with Galois group of derived length at most m. The Galois group $\mathrm{G}_p^m(K) = \mathrm{Gal}(\mathrm{F}_p^m(K)|K)$ is called the mth *p-class group* of K [10].

$\mathrm{F}_p^\infty(K)$ maximal unramified pro-p extension of K,
$\mathrm{G}_p^\infty(K) = \mathrm{Gal}(\mathrm{F}_p^\infty(K)|K)$ the *p-tower group* of K,
$\ell = \ell_p(K)$ *length* of the p-class field tower of K,
$K < \mathrm{F}_p^1(K) < \mathrm{F}_p^2(K) < \ldots < \mathrm{F}_p^{\ell-1}(K) < \mathrm{F}_p^\ell(K) = \mathrm{F}_p^\infty(K)$.

[10] D. C. Mayer, The distribution of second p-class groups on coclass graphs, *J. Théor. Nombres Bordeaux* **25** (2013), no. 2, 401–456.

§ 1.1. Known Length of p-Towers

(1) $\ell_p(K) = 0$ if and only if $\mathrm{r}_p(K) = 0$.
(2) For any *quadratic* field $K = \mathbb{Q}(\sqrt{D})$,
$\ell_p(K) = 1$ if and only if $\mathrm{r}_p(K) = 1$.

Theorem 1.1. (Golod & Shafarevich [1964], Vinberg, Gaschütz, Koch & Venkov [1975] [8])
For *complex quadratic* fields $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, the condition $\mathrm{r}_p(K) \geq 3$ implies $\ell_p(K) = \infty$.

[8] H. Koch und B. B. Venkov, Über den p-Klassenkörperturm eines imaginär-quadratischen Zahlkörpers, *Astérisque* **24–25** (1975), 57–67.

Open Problem, Motivation for our Research.

For a quadratic field K with $\mathrm{r}_p(K) = 2$, the entire range $2 \leq \ell_p(K) \leq \infty$ seems to be admissible.

However, till August 2012, the exact length $\ell_p(K)$ for complex quadratic fields K with $\mathrm{r}_p(K) = 2$ and $p \in \{3, 5, 7\}$ was known for two-stage towers, $\ell_p(K) = 2$, only.

For $p = 3$, $\mathrm{Cl}_3(K) \simeq (3, 3)$, resp. $(3, 9)$, two-stage towers occur with relative frequency $\frac{936}{2020} \approx 46.3\%$, resp. $\frac{406}{875} \approx 46.4\%$.

Theorem 1.2.
(Boston, Bush & Mayer [August 24, 2012])
There exist complex quadratic fields K
with $\mathrm{Cl}_3(K) \simeq (3,3)$ and $\ell_3(K) = 3$.

Our aim is to give a new proof
of a more precise assertion.

Theorem 1.3. (Mayer [May 05, 2013])
There exist complex quadratic fields K
with $\mathrm{Cl}_3(K) \simeq (3,9)$ and $\ell_3(K) = 3$.

§ 1.2. σ-Groups and Schur Groups

Definition. A pro-p group G is called a *σ-group*, if it admits an automorphism $\sigma \in \mathrm{Aut}(G)$ acting as inversion $x \mapsto x^{-1}$ on the abelianization G/G'.

Theorem 1.4. (Artin [1928] [7])
For any *quadratic* field K, the p-tower group $\mathrm{G}_p^{\infty}(K)$ and the groups $\mathrm{G}_p^{n}(K)$, $n \geq 2$, are σ-groups.

[7] G. Frei, P. Roquette, and F. Lemmermeyer, *Emil Artin and Helmut Hasse. Their Correspondence 1923–1934*, Universitätsverlag Göttingen, 2008.

G a pro-p group,
$d(G) = \dim_{\mathbb{F}_p}(\mathrm{H}^1(G, \mathbb{F}_p))$ the *generator rank* of G,
$r(G) = \dim_{\mathbb{F}_p}(\mathrm{H}^2(G, \mathbb{F}_p))$ the *relation rank* of G.

Definition. A pro-p group G which satisfies the equation $r(G) = d(G)$ is said to have a *balanced presentation*, or to be a *Schur group*.

Theorem 1.5. (Shafarevich [1963] [15])
The p-tower group $\mathrm{G}_p^\infty(K)$ of a *complex quadratic* field $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, is a Schur group.

[15] I. R. Shafarevich, Extensions with prescribed ramification points, *Publ. Math., Inst. Hautes Études Sci.* **18** (1963), 71–95 (Russian). English transl. by J. W. S. Cassels: *Am. Math. Soc. Transl.*, II. Ser., **59** (1966), 128–149.

For $m \in \mathbb{N} \cup \{\infty\}$, we have $d(\mathrm{G}_p^m(K)) = \mathrm{r}_p(K)$.

§ 1.3. Cover and Schur Cover

Definition. The *cover*, $\mathrm{cov}(G)$, of a finite metabelian p-group G is defined as the set of all (isomorphism classes of) finite non-metabelian p-groups H whose second derived quotient, that is the metabelianization, H/H'', is isomorphic to G. The subset of the cover, $\mathrm{cov}(G)$, consisting of Schur groups is called the *Schur cover*, $\mathrm{cov}_*(G)$, of G.

Theorem 1.6. (Bartholdi & Bush [2007] [3]) There exist metabelian 3-groups of coclass 2 with infinite Schur cover.

For $\mathrm{Cl}_3(K) \simeq (3,3)$,
they occur with relative frequency $\frac{297}{2020} \approx 14.7\%$.

[3] L. Bartholdi and M. R. Bush, Maximal unramified 3-extensions of imaginary quadratic fields and $\mathrm{SL}_2\mathbb{Z}_3$, *J. Number Theory* **124** (2007), 159–166.

We focus on searching for metabelian 3-groups G which have a unique Schur cover, $\#\mathrm{cov}_*(G) = 1$.

§ 2. Sieving p-groups in the generation algorithm

§ 2.1. Transfer Kernel Type

Definition.

G a p-group of generator rank $d(G) = 2$,
$H_1, \ldots, H_{p+1}$ its maximal subgroups,
$T_i : G/G' \to H_i/H_i'$, for $1 \le i \le p+1$,
the *Artin transfers* from G to the H_i [2].

The family $\varkappa(G) = (\ker(T_i))_{1 \le i \le p+1}$
is called the *transfer kernel type* (TKT) of G.

[2] E. Artin, Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz, *Abh. Math. Sem. Univ. Hamburg* **7** (1929), 46–51.

We shall be concerned with $p = 3$ and the following TKTs [9].

1. Two closely related sections,
inseparably associated with each other,
which we want to filter by sieving:

- All four cases of TKTs in section E, that is,
  E.6, $\varkappa = (1, 1, 2, 2)$,
  E.14, $\varkappa = (3, 1, 2, 2) \sim (4, 1, 2, 2)$,
  E.8, $\varkappa = (1, 1, 3, 4)$, and
  E.9, $\varkappa = (3, 1, 3, 4) \sim (4, 1, 3, 4)$.
- Both TKTs in section c, that is,
  c.18, $\varkappa = (0, 1, 2, 2)$, and
  c.21, $\varkappa = (0, 1, 3, 4)$.

2. TKTs we are going to ignore,
since they disturb the intended structure:

- TKT H.4, $\varkappa = (2, 1, 2, 2)$.
- TKT G.16, $\varkappa = (2, 1, 3, 4)$.

[9] D. C. Mayer, Transfers of metabelian p-groups, *Monatsh. Math.* **166** (2012), no. 3–4, 467–495.

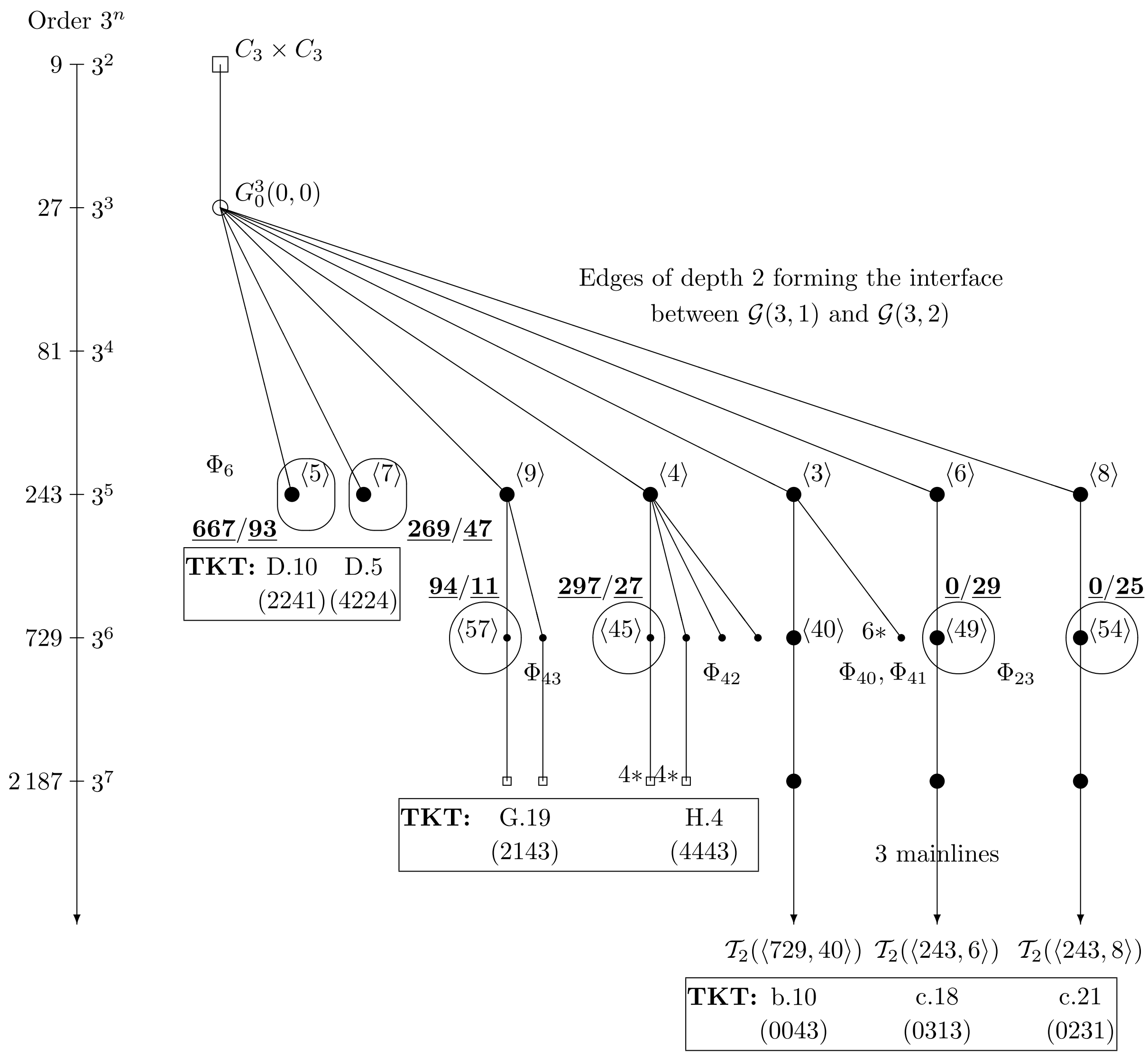
Number Theoretic Main Conjecture.
(Mayer and Newman [2013])
If the TKT $\varkappa(G)$ of the second 3-class group $G = \mathrm{G}_3^2(K)$ of a *complex quadratic* number field $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, with $\mathrm{Cl}_3(K) \simeq (3,3)$ belongs to the four types of section E, then the 3-tower of K has exactly three stages, that is, $\ell_3(K) = 3$.

Among complex quadratic fields K with $\mathrm{Cl}_3(K) \simeq (3,3)$, those with TKT in section E occur with relative frequency $\frac{411}{2020} \approx 20.3\%$.

Corresponding complex quadratic fields K with $\mathrm{Cl}_3(K) \simeq (3,9)$, occur with relative frequency $\frac{182}{875} \approx 20.8\%$.

In the sequel we characterize 3-groups by their identifer in the *SmallGroups library* and their descendants of order bigger than 3^7 by the notation used in the ANUPQ package of GAP and MAGMA.

Figure 1. Distribution of 2020/2576 Groups $G_3^2(K)$ on the Coclass Graph $\mathcal{G}(3,2)$



§ 2.2. Location of 3-groups with TKT E

Theorem 2.1. (Mayer [2013])
G a 3-group with TKT in section c or E.

(1) G non-metabelian $\Longrightarrow$
either $G \in \mathcal{T}(\langle 243, 6\rangle)$ or $G \in \mathcal{T}(\langle 243, 8\rangle)$,
and G can be of any coclass $\mathrm{cc}(G) \geq 2$.

(2) G metabelian $\Longrightarrow \mathrm{cc}(G) = 2$,
and either $G \in \mathcal{T}_2(\langle 243, 6\rangle)$ or $G \in \mathcal{T}_2(\langle 243, 8\rangle)$.

Conjecture 2.1.
G non-metabelian $\Longrightarrow G'' \leq \zeta_1(G)$, that is,
G is centre-by-metabelian, and thus $\mathrm{dl}(G) = 3$.

FIGURE 2. Distribution of 2020/2576 Groups $G_3^2(K)$ on the Coclass Tree $\mathcal{T}_2(\langle 243,6\rangle)$

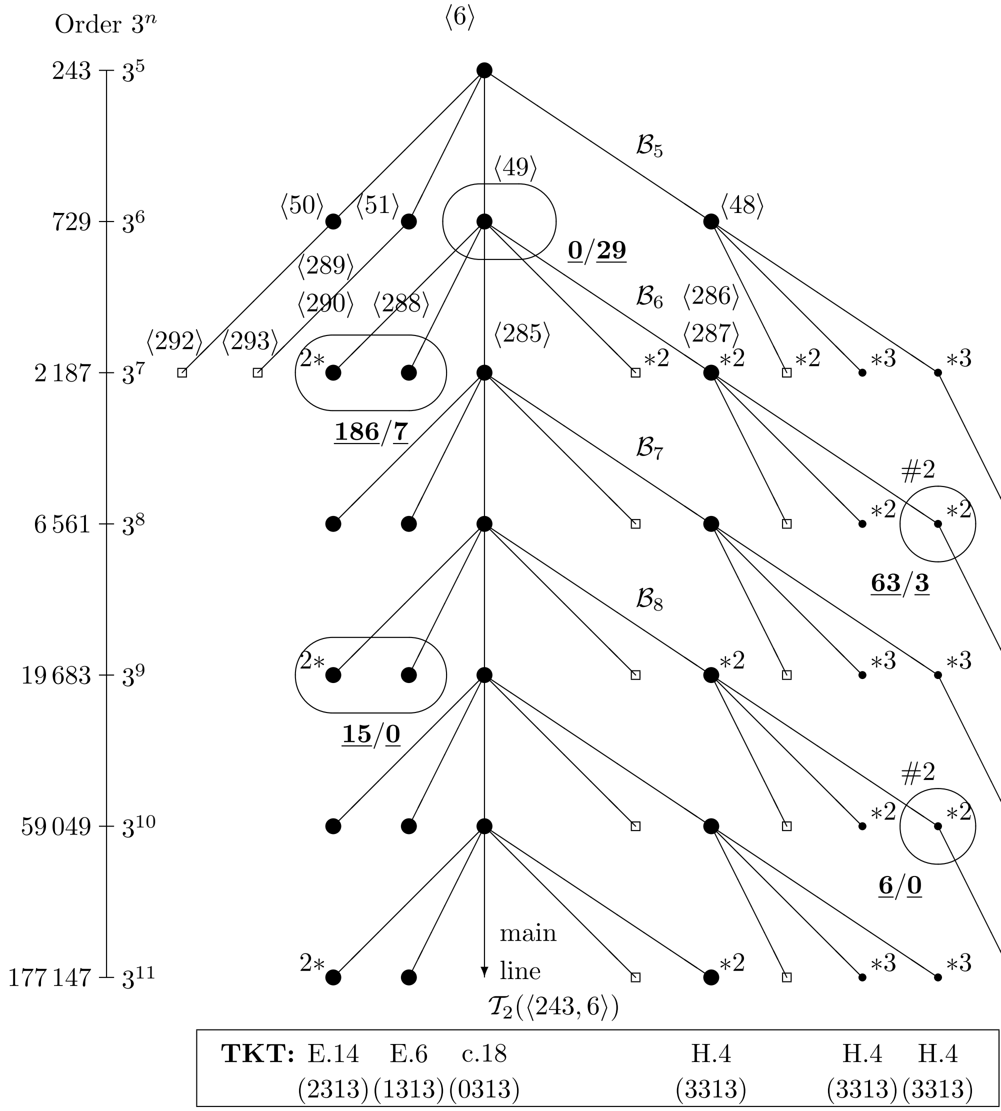
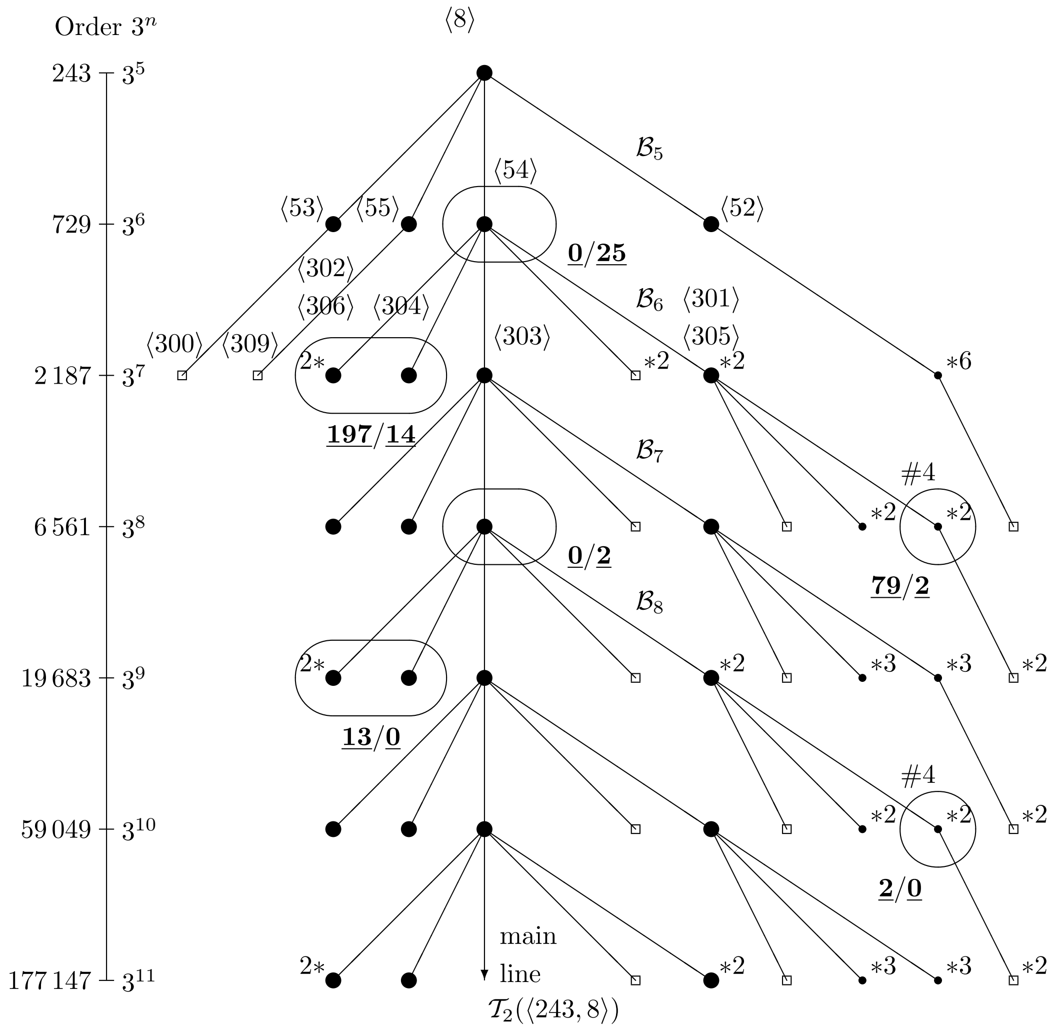


FIGURE 3. Distribution of 2020/2576 Groups $G_3^2(K)$ on the Coclass Tree $\mathcal{T}_2(\langle 243, 8\rangle)$



TKT:	E.9	E.8	c.21	G.16	G.16	G.16
	(2231)	(1231)	(0231)	(4231)	(4231)	(4231)

Definition. The restrictions of the *descendant tree* $\mathcal{T}(G)$ [13], resp. the coclass subtree $\mathcal{T}_r(G)$ [10,12], of a finite p-group G of coclass $\mathrm{cc}(G) = r \geq 1$ to vertices of assigned TKTs are denoted by $\mathcal{T}^*(G)$, resp. $\mathcal{T}_r^*(G)$, and are called TKT-*pruned trees* with respect to the assigned TKTs.

In the sequel, we TKT-prune all trees with respect to sections c and E (sections H and G are cancelled).

The following two figures are drawn independently from a concrete realization of the starting group $G_0^{(0)}$ which is assumed to be of order 3^n and coclass r. They visualize a *multiple periodicity* of the TKT-pruned descendant tree $\mathcal{T}^*(G_0^{(0)})$.

- Firstly, the *well-known periodicity of (depth-) pruned branches* of depth 1 of all coclass subtrees $\mathcal{T}_{r+i}(G_0^{(i)})$ with $i \geq 0$, according to du Sautoy [4], resp. to Eick & Leedham-Green [5].
- Secondly, a *new periodicity of bifurcations* at immediate mainline descendants $G_1^{(i)}$ of subtree roots $G_0^{(i)}$ which are not coclass-settled, and a new periodicity of TKT-pruned coclass subtrees $\mathcal{T}_{r+i}^*(G_0^{(i)})$ with $i \geq 0$, isomorphic as graphs.

FIGURE 4. Multi-periodic TKT-pruned descendant tree $\mathcal{T}^*(G_0^{(0)})$ restricted to σ-groups.

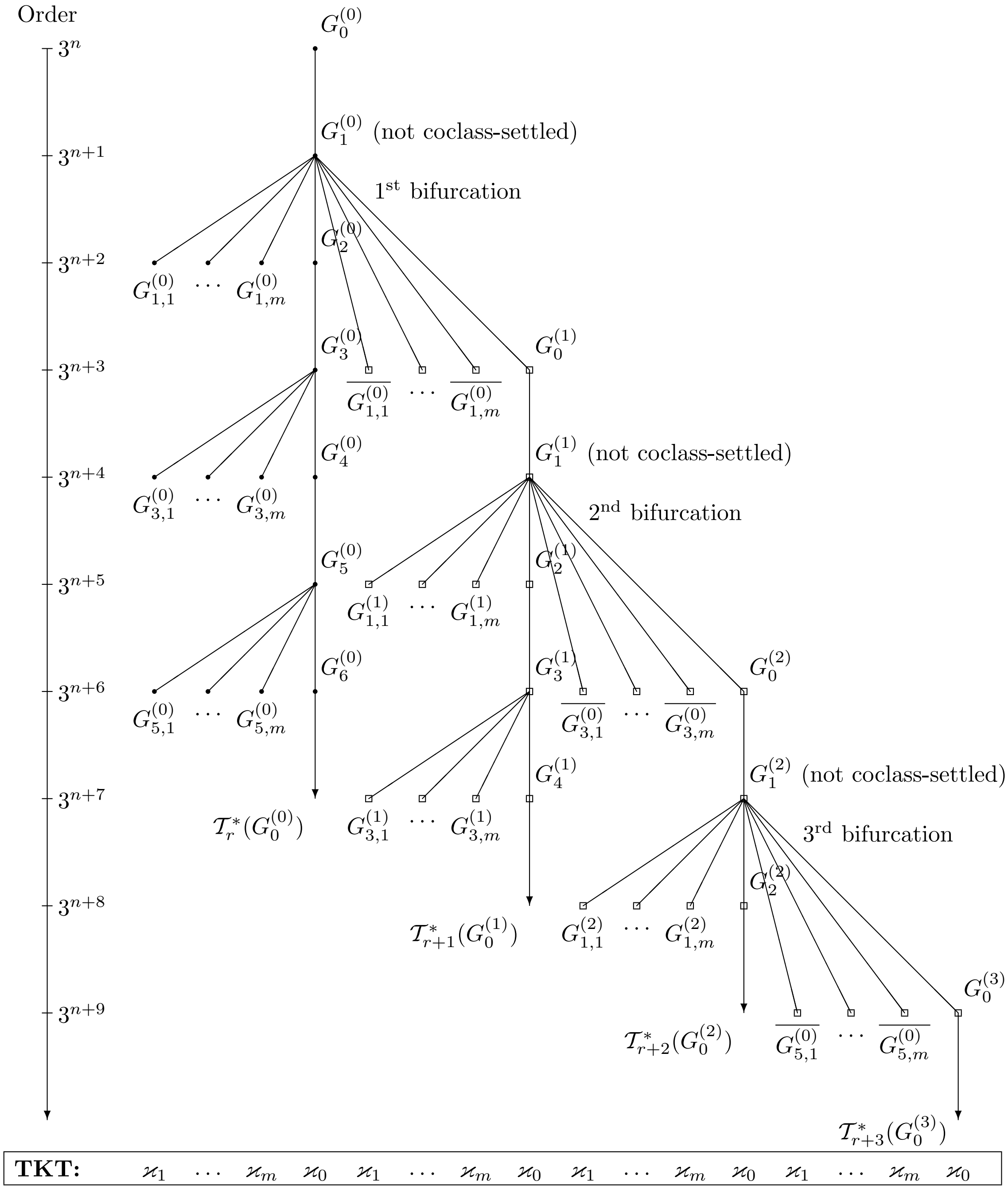
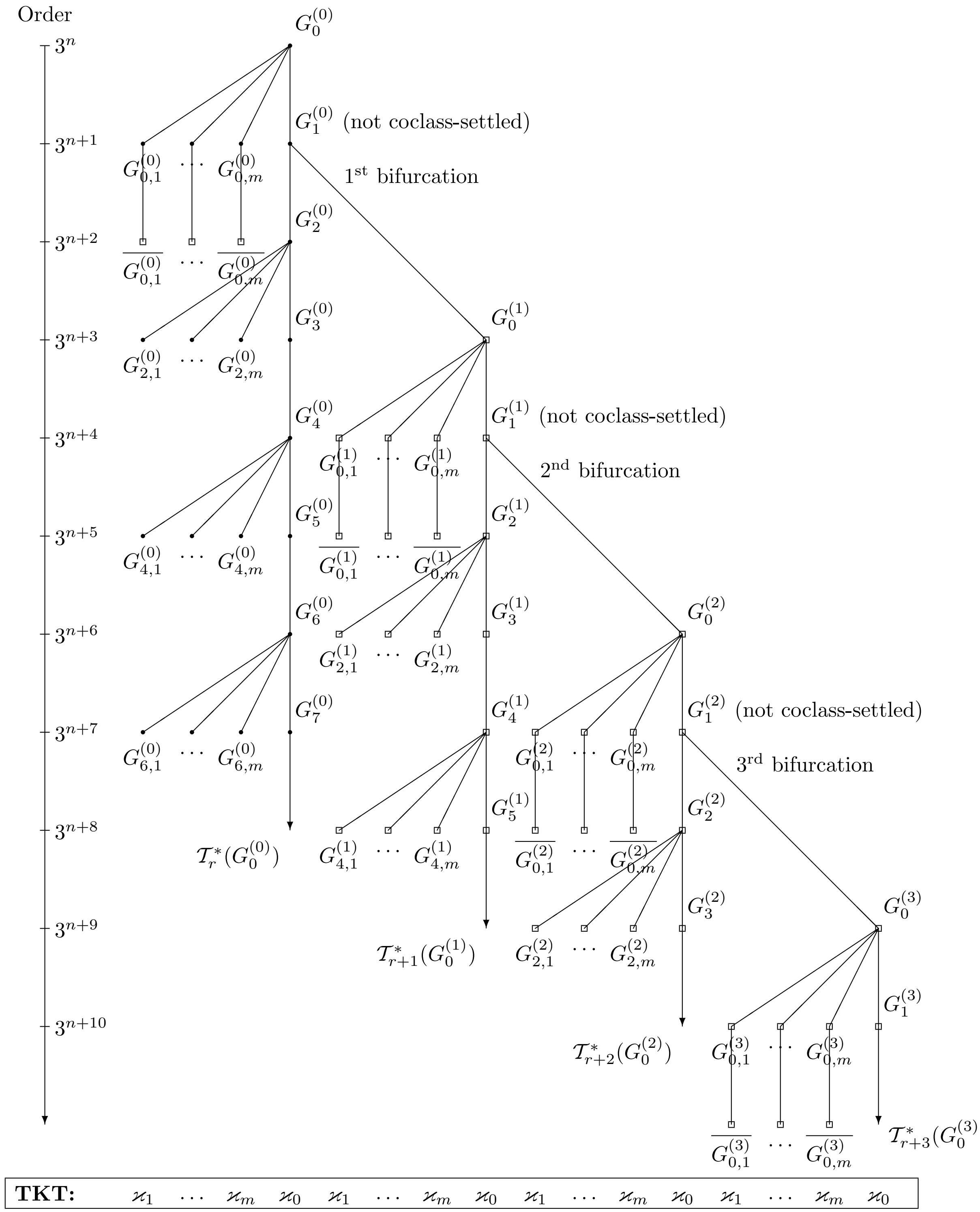


FIGURE 5. Multi-periodic TKT-pruned descendant tree $\mathcal{T}^*(G_0^{(0)})$ restr. to non-σ groups.



Group Theoretic Main Conjecture.
(Mayer and Newman [2013])
The cardinalities of the covers and Schur covers of all vertices of depth 1 on the TKT-pruned coclass tree $\mathcal{T}_2^*(G_0^{(0)})$ with root $G_0^{(0)}$ either $\langle 243, 6\rangle$ or $\langle 243, 8\rangle$ are finite. They are given by

(1) $\#\mathrm{cov}(G_{2\ell+1,k}^{(0)}) = \ell+1$ and $\#\mathrm{cov}_*(G_{2\ell+1,k}^{(0)}) = 1$, for $\ell \geq 0$ and $1 \leq k \leq 3$, that is, σ-groups have a unique Schur σ-group as their Schur cover,

(2) $\#\mathrm{cov}(G_{2\ell,k}^{(0)}) = \ell + 1$ and $\#\mathrm{cov}_*(G_{2\ell,k}^{(0)}) = 0$, for $\ell \geq 0$ and $1 \leq k \leq 2$, that is, the Schur cover of non-σ groups is empty.

§ 3. Details of the Proof

§ 3.0. Group Theory Notation

$p \geq 3$ an odd prime,
G a finite p-group of order p^n, $n \geq 1$,

$\gamma_j(G)$, $j \geq 1$, terms of *lower* central series of G,
$G = \gamma_1(G) > \gamma_2(G) = G' > \ldots > \gamma_c(G) > \gamma_{c+1}(G) = 1$,
$c = \mathrm{cl}(G)$ the *class* of G,
$\mathrm{cc}(G)$ the *coclass* of G, such that $n = \mathrm{cl}(G) + \mathrm{cc}(G)$,

$\zeta_j(G)$, $j \geq 0$, terms of *upper* central series of G,
$1 = \zeta_0(G) < \zeta_1(G) = Z(G) < \ldots < \zeta_{c-1}(G) < \zeta_c(G) = G$,

$G^{(j)}$, $j \geq 0$, terms of the *derived* series of G,
$G = G^{(0)} > G^{(1)} = G' > G^{(2)} = G'' > \ldots > G^{(\ell-1)} > G^{(\ell)} = 1$,
$\ell = \mathrm{dl}(G)$ the *derived length* of G.

Blackburn's *two-step centralizers*:
$\chi_j(G)/\gamma_{j+2}(G)$ the centralizer of $\gamma_j(G)/\gamma_{j+2}(G)$,
for $j \geq 2$, that is,
the biggest subgroup $G' \leq \chi_j(G) \leq G$ such that
$[\chi_j(G), \gamma_j(G)] \leq \gamma_{j+2}(G)$.

$x, y \in G$, $n \geq 0$,

$[y, x]^{(n)} = [y, \overbrace{x, \ldots, x}^{n \text{ times}}]$ the nth *Engel commutator* of y with respect to x.

§ 3.1. Further Group Theory Notation

G a finite p-group of p-class $\mathrm{cl}_p(G) = c$, where
$P_j(G)$, $j \ge 0$, terms of *lower* p-central series of G,
$G = P_0(G) > P_1(G) = G^pG' > \ldots > P_{c-1}(G) > P_c(G) = 1$,
F free pro-p group with $d(G)$ generators,
$1 \to R \to F \to G \to 1$, presentation for
$G \simeq F/R$,
R^* the topological closure of $R^p[F,R] \le R$,
$G^* = F/R^*$ the *p-covering group* of G,
R/R^* the *p-multiplicator* of G (elementary abelian),
$\mathrm{MR}(G) = \dim_{\mathbb{F}_p}(R/R^*)$ *multiplicator rank* of G,
$P_c(G^*)$ the *nucleus* of G,
$\mathrm{NR}(G) = \dim_{\mathbb{F}_p}(P_c(G^*))$ the *nuclear rank* of G [14].

Remarks.
$\mathrm{NR}(G) = 0$ if and only if G is a *terminal* vertex.
$\mathrm{NR}(G) \ge 1$ if and only if G is a *capable* vertex.
If $\mathrm{NR}(G) \ge 2$, then G is *not coclass-settled*,
that is, $\mathcal{T}(G) \not\subseteq \mathcal{G}(p,r)$ for $r = \mathrm{cc}(G)$.
G is a Schur group if and only if
$\mathrm{MR}(G) = d(G)$ and $\mathrm{NR}(G) = 0$.

§ 3.2. New Metabelian Parametrized PC-Presentations

The following result shows that certain 3-groups of class at least 5 on the coclass tree $\mathcal{T}_2(\langle 243, 6\rangle)$ with metabelian mainline belong to $6+4=10$ periodic coclass sequences with period length 2.

Theorem 3.1. (Mayer [2013])
For each integer $c \geq 5$, there are 6 metabelian descendants G of $\langle 243, 6\rangle$, having nilpotency class $\mathrm{cl}(G)=c$, coclass $\mathrm{cc}(G)=2$, and order $|G|=3^{c+2}$, with two generators x, y and parametrized pc-presentation

$$\begin{aligned} G = \langle\ & x, y, s_2, t_3, s_3, s_4, \ldots, s_c \ \mid \\ & s_2=[y,x],\ t_3=[s_2,y],\ s_j=[s_{j-1},x] \text{ for } 3 \leq j \leq c, \\ & s_j^3 = s_{j+2}^2 s_{j+3} \text{ for } 2 \leq j \leq c-3,\ s_{c-2}^3 = s_c^2,\ t_3^3=1, \\ & R(x)=1,\ R(y)=1\ \rangle, \end{aligned}$$

where the relators $R(x)$ and $R(y)$ are given by

$$(1)\ R(x) = \begin{cases} x^3 & \text{for } G \text{ of TKT c.18 or H.4,} \\ x^3 s_c^{-1} & \text{for } G \text{ of TKT E.6 or E.14,} \end{cases}$$

$$(2)\ R(y) = \begin{cases} y^3 s_3^{-2} s_4^{-1} & \text{for } G \text{ of TKT c.18 or E.6,} \\ y^3 s_3^{-2} s_4^{-1} s_c^{-1} & \text{or} \\ y^3 s_3^{-2} s_4^{-1} s_c^{-2} & \text{for } G \text{ of TKT H.4 or E.14.} \end{cases}$$

For odd class $c \geq 5$ the 6 groups are pairwise non-isomorphic σ-groups.
For even class $c \geq 6$, the two pairs of groups sharing the same TKT (H.4 and E.14) are isomorphic, and thus only 4 of these non-σ groups are pairwise non-isomorphic.
None of the groups is a Schur group.

Corollary 3.1.1. (Nebelung [1989] [11])
For each $c \geq 5$, the factors of the lower and upper central series of all groups in Theorem 3.1 are given by

$$\gamma_j(G)/\gamma_{j+1}(G) \simeq \begin{cases}(3,3) & \text{for } j \in \{1,3\},\\ (3) & \text{for } j = 2 \text{ or } 4 \leq j \leq c,\end{cases}$$
$$\zeta_j(G)/\zeta_{j-1}(G) \simeq \begin{cases}(3,3) & \text{for } j \in \{1,c\},\\ (3) & \text{for } 2 \leq j \leq c-1.\end{cases}$$

The two-step centralizers form a *monotonic* chain,
$G' = \chi_2(G) < \chi_3(G) = \ldots = \chi_{c-1}(G) = H_1 < \chi_c(G) = G.$

Corollary 3.1.2. (Mayer [2013])
For each $c \geq 5$, the Artin transfers

$$T_i:\ G/G' \to H_i/H_i',\ gG' \mapsto \begin{cases} g^3 H_i' & \text{if } g \in G \setminus H_i,\\ g^{S_3(h)} H_i' & \text{if } g \in H_i,\end{cases}$$

where the commutator groups of the maximal subgroups are

$$\begin{aligned} H_1' &= \langle t_3 \rangle,\\ H_2' &= \langle s_3, s_4, \ldots, s_c \rangle,\\ H_3' &= \langle s_3 t_3, s_4, \ldots, s_c \rangle,\\ H_4' &= \langle s_3 t_3^2, s_4, \ldots, s_c \rangle,\end{aligned}$$

and $S_3(h) = 1 + h + h^2$, are given by the images

$$T_1(x^j y^\ell G') \equiv \begin{cases} s_c^{e\ell} \mod H_1' & \text{if } x^3 = 1,\ y^3 = s_3^2 s_4 s_c^e,\\ s_c^{j+e\ell} \mod H_1' & \text{if } x^3 = s_c,\ y^3 = s_3^2 s_4 s_c^e,\end{cases}$$
$$T_2(x^j y^\ell G') \equiv t_3^{-j} \mod H_2',$$
$$T_i(x^j y^\ell G') \equiv s_3^{2\ell} \mod H_i' \text{ for } i \in \{3,4\},$$

where $-1 \leq j, \ell \leq 1$ and $0 \leq e \leq 2$.

§ 3.3. The Metabelian Limit

Theorem 3.2.

(Eick, Leedham-Green, Newman, O'Brien [2011] [6]) The projective limit $L = \varprojlim_{j\ge 0} G_j^{(0)}$ of the metabelian mainline $(G_j^{(0)})_{j\ge 0}$ of the coclass tree $\mathcal{T}_2(G_0^{(0)})$ with root $G_0^{(0)} = \langle 243, 6\rangle$, resp. $\langle 243, 8\rangle$, is given by the pro-3 presentation

$$L = \langle\ t, a, z \ \mid\ a^3 = z^f,\ [t, t^a] = z,\ tt^a t^{a^2} = z^2,\ z^3 = 1,\ [z, a] = 1,\ [z, t] = 1,\ \rangle,$$

where $f = 0$, resp. 1. The centre of L is the cyclic group $\zeta_1(L) = \langle z\rangle$ of order 3.

Corollary 3.2.

The mainline vertices of $\mathcal{T}_2(G_0^{(0)})$ are the σ-groups

$$\begin{aligned} G_{2\ell}^{(0)} &\simeq L/\langle t^{3^{\ell+2}}\rangle \\ &\quad \text{of order } 3^{2\ell+5} \text{ and odd class } 2\ell+3, \\ G_{2\ell+1}^{(0)} &\simeq L/\langle t^{3^{\ell+2}}, t^{3^{\ell+1}}(t^a)^{-3^{\ell+1}}\rangle \\ &\quad \text{of order } 3^{2\ell+6} \text{ and even class } 2\ell+4, \end{aligned}$$

for $\ell \ge 0$.

§ 3.4. First Non-Metabelian Parametrized PC-Presentations

The following result shows that certain 3-groups of class at least 6 on the entirely non-metabelian coclass tree $\mathcal{T}_3(\langle 729, 49\rangle - \#2; 1)$ belong to $6 + 4 = 10$ periodic coclass sequences with period length 2.

Theorem 3.3. (Mayer [2013])
For each integer $c \geq 6$, there are 6 descendants G of $\langle 729, 49\rangle - \#2; 1$, having nilpotency class $\mathrm{cl}(G) = c$, coclass $\mathrm{cc}(G) = 3$, order $|G| = 3^{c+3}$, and derived length $\mathrm{dl}(G) = 3$, with two generators x, y and parametrized pc-presentation

$$\begin{aligned} G \ = \ \langle\ & x, y, s_2, t_3, s_3, s_4, \ldots, s_c, u_5 \ \mid \\ & s_2 = [y, x],\ t_3 = [s_2, y],\ s_j = [s_{j-1}, x] \text{ for } 3 \leq j \leq c, \\ & u_5 = [s_3, y] = [s_4, y],\ [s_3, s_2] = u_5^2,\ t_3^3 = u_5^2, \\ & s_2^3 = s_4^2 s_5 u_5,\ s_j^3 = s_{j+2}^2 s_{j+3} \text{ for } 3 \leq j \leq c-3,\ s_{c-2}^3 = s_c^2, \\ & R(x) = 1,\ R(y) = 1\ \rangle, \end{aligned}$$

where the relators $R(x)$ and $R(y)$ are given by equations (1) and (2).
For odd class $c \geq 7$ the 6 groups are pairwise non-isomorphic σ-groups.
For even class $c \geq 6$, the two pairs of groups sharing the same TKT (H.4 and E.14) are isomorphic, and thus only 4 of these non-σ groups are pairwise non-isomorphic.

Corollary 3.3.1. (Mayer [2013])
For each $c \geq 5$, the factors of the lower and upper central series of all groups in Theorem 3.3 are given by

$$\gamma_j(G)/\gamma_{j+1}(G) \simeq \begin{cases} (3,3) & \text{for } j \in \{1,3,5\}, \\ (3) & \text{for } j \in \{2,4\} \text{ or } 6 \leq j \leq c, \end{cases}$$

$$\zeta_j(G)/\zeta_{j-1}(G) \simeq \begin{cases} (3,9) & \text{for } j = 1, \\ (3) & \text{for } 2 \leq j \leq c-1, \\ (3,3) & \text{for } j = c. \end{cases}$$

The chain of two-step centralizers is *not monotonic*,
$G' = \chi_2(G) < \chi_3(G) = H_1 > \chi_4(G) = G' <$
$< \chi_5(G) = \ldots = \chi_{c-1}(G) = H_1 < \chi_c(G) = G$.

Sporadic siblings of $\langle 729, 49\rangle - \#2; 1$

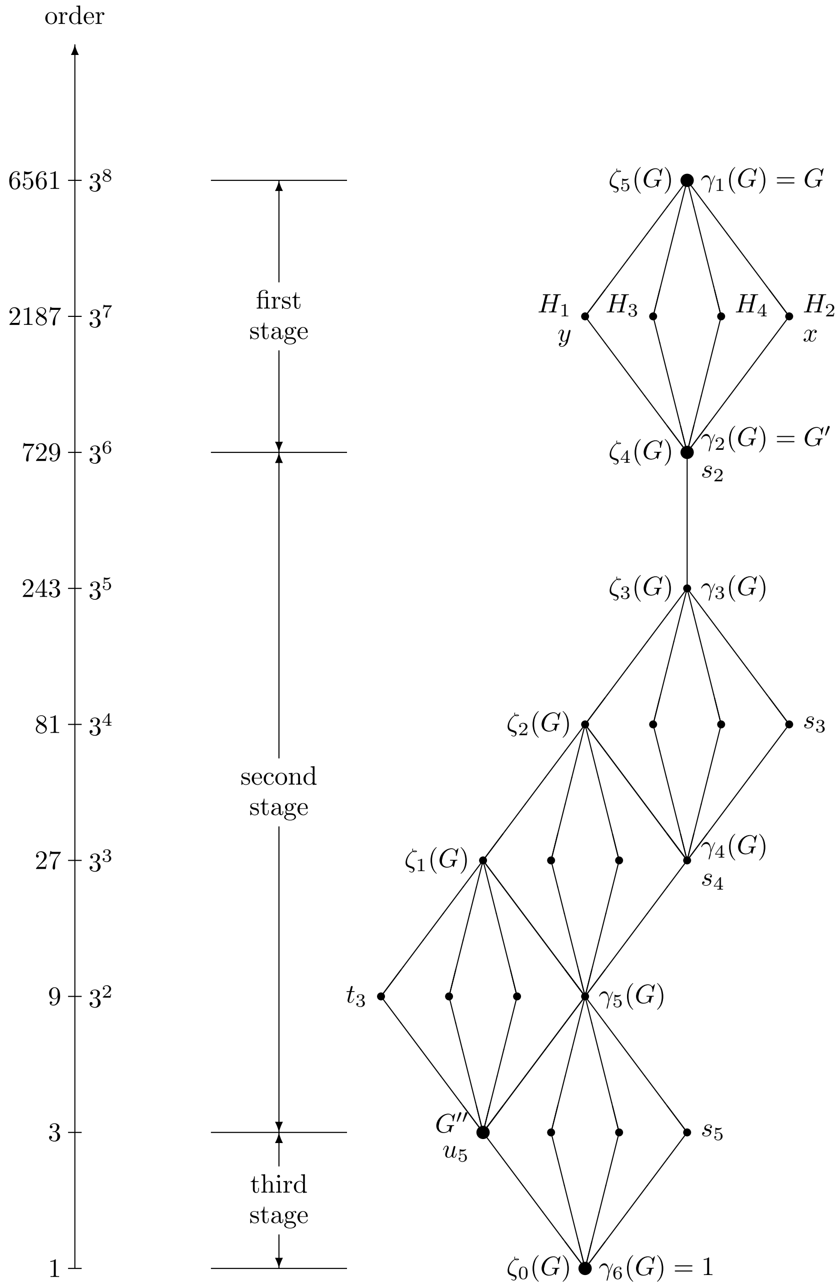
We show that there exist three non-metabelian 3-groups of class 5 which are isolated siblings of $\langle 729, 49\rangle - \#2; 1$ and form unique Schur covers of the three unbalanced metabelian 3-groups with TKT in section E and class 5 on the coclass tree $\mathcal{T}_2(\langle 243, 6\rangle)$.

Corollary 3.3.2. (Mayer [2013])
There are 6 immediate descendants G of depth 2 of $\langle 729, 49\rangle$, having nilpotency class $\mathrm{cl}(G) = 5$, coclass $\mathrm{cc}(G) = 3$, order $|G| = 3^8$, and derived length $\mathrm{dl}(G) = 3$, with two generators x, y and pc-presentation

$$\begin{aligned} G \;=\; \langle\; & x, y, s_2, t_3, s_3, s_4, s_5, u_5 \;\mid \\ & s_2 = [y, x],\; t_3 = [s_2, y],\; s_j = [s_{j-1}, x] \text{ for } 3 \le j \le 5, \\ & u_5 = [s_3, y] = [s_4, y],\; [s_3, s_2] = u_5^2,\; t_3^3 = u_5^2, \\ & s_2^3 = s_4^2 s_5 u_5,\; s_3^3 = s_5^2,\; R(x) = 1,\; R(y) = 1 \;\rangle, \end{aligned}$$

where the relators $R(x)$ and $R(y)$ are given by equations (1) and (2) with $c = 5$.
The 3 isolated vertices $\langle 729, 49\rangle - \#2; 4$ of TKT E.6 and $\langle 729, 49\rangle - \#2; 5$, $\langle 729, 49\rangle - \#2; 6$ of TKT E.14 among the 6 descendants are Schur σ-groups.

Figure 6. Full normal lattice, including upper and lower central series, of a 3-group G with $G/G' \simeq (3,3)$, $|G| = 3^8$, $\mathrm{cl}(G) = 5$, $\mathrm{cc}(G) = 3$, $\mathrm{dl}(G) = 3$, satisfying $R(x) = 1$ and $R(y) = 1$ with relators $R(x), R(y)$ given by equations (1) and (2)



The following result shows that certain 3-groups of class at least 8 on the entirely non-metabelian coclass tree $\mathcal{T}_4(\langle 729, 49\rangle - \#2; 1 - \#1; 1 - \#2; 1)$ belong to $6 + 4 = 10$ periodic coclass sequences with period length 2.

Theorem 3.4. (Mayer [2013])
For each integer $c \geq 8$, there are 6 descendants G of $\langle 729, 49\rangle - \#2; 1 - \#1; 1 - \#2; 1$, having nilpotency class $\mathrm{cl}(G) = c$, coclass $\mathrm{cc}(G) = 4$, order $|G| = 3^{c+4}$, and derived length $\mathrm{dl}(G) = 3$, with two generators x, y and parametrized pc-presentation

$$\begin{aligned} G \;=\; \langle\; & x, y, s_2, t_3, s_3, s_4, \ldots, s_c, u_5, u_7 \;\; | \\ & s_2 = [y, x],\; t_3 = [s_2, y],\; s_j = [s_{j-1}, x] \text{ for } 3 \leq j \leq c, \\ & u_5 = [s_4, y],\; u_7 = [s_6, y],\; [s_3, s_2] = u_5^2 u_7^2,\; [s_3, y] = u_5 u_7^2, \\ & [s_5, y] = u_7^2,\; [s_4, s_2] = u_7^2,\; [s_5, s_2] = u_7^2,\; [s_4, s_3] = u_7, \\ & s_2^3 = s_4^2 s_5 u_5,\; s_3^3 = s_5^2 s_6 u_7^2,\; t_3^3 = u_5^2 u_7^2,\; u_5^3 = u_7^2, \\ & s_j^3 = s_{j+2}^2 s_{j+3} \text{ for } 4 \leq j \leq c-3,\; s_{c-2}^3 = s_c^2, \\ & R(x) = 1,\; R(y) = 1 \;\rangle, \end{aligned}$$

where the relators $R(x)$ and $R(y)$ are given by equations (1) and (2).
For odd class $c \geq 9$ the 6 groups are pairwise non-isomorphic σ-groups.
For even class $c \geq 8$, the two pairs of groups sharing the same TKT (H.4 and E.14) are isomorphic, and thus only 4 of these non-σ groups are pairwise non-isomorphic.

Corollary 3.4.1. (Mayer [2013])
For each $c \geq 7$, the factors of the lower and upper central series of all groups in Theorem 3.4 are given by

$$\gamma_j(G)/\gamma_{j+1}(G) \simeq \begin{cases} (3,3) & \text{for } j \in \{1,3,5,7\}, \\ (3) & \text{for } j \in \{2,4,6\} \text{ or } 8 \leq j \leq c, \end{cases}$$

$$\zeta_j(G)/\zeta_{j-1}(G) \simeq \begin{cases} (3,27) & \text{for } j = 1, \\ (3) & \text{for } 2 \leq j \leq c-1, \\ (3,3) & \text{for } j = c. \end{cases}$$

The chain of two-step centralizers is *not monotonic*,
$G' = \chi_2(G) < \chi_3(G) = H_1 > \chi_4(G) = G' <$
$< \chi_5(G) = H_1 > \chi_6(G) = G' <$
$< \chi_7(G) = \ldots = \chi_{c-1}(G) = H_1 < \chi_c(G) = G$.

Sporadic siblings of $\langle 729, 49\rangle - \#2; 1 - \#1; 1 - \#2; 1$

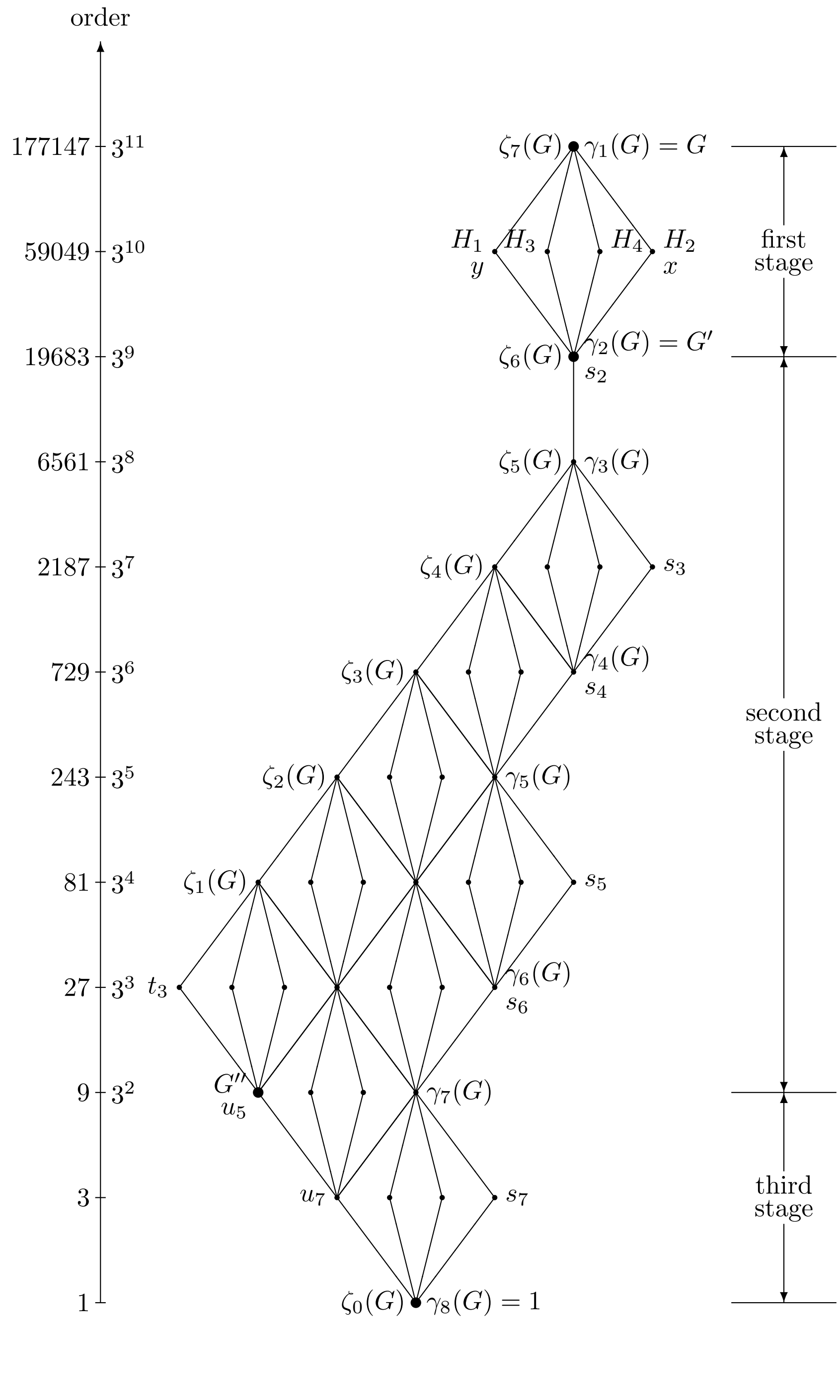
We show that there exist three non-metabelian 3-groups of class 7 which are isolated siblings of $\langle 729, 49\rangle - \#2; 1 - \#1; 1 - \#2; 1$ and form unique Schur covers of the three unbalanced metabelian 3-groups with TKT in section E and class 7 on the coclass tree $\mathcal{T}_2(\langle 243, 6\rangle)$.

Corollary 3.4.2. (Mayer [2013])
There are 6 immediate descendants G of depth 2 of $\langle 729, 49\rangle - \#2; 1 - \#1; 1$, having nilpotency class $\mathrm{cl}(G) = 7$, coclass $\mathrm{cc}(G) = 4$, order $|G| = 3^{11}$, and derived length $\mathrm{dl}(G) = 3$, with two generators x, y and pc-presentation

$$\begin{aligned} G \;=\; \langle\; & x, y, s_2, t_3, s_3, s_4, \ldots, s_7, u_5, u_7 \;\mid \\ & s_2 = [y, x],\; t_3 = [s_2, y],\; s_j = [s_{j-1}, x] \text{ for } 3 \le j \le 7, \\ & u_5 = [s_4, y],\; u_7 = [s_6, y],\; [s_3, s_2] = u_5^2 u_7^2,\; [s_3, y] = u_5 u_7^2, \\ & [s_5, y] = u_7^2,\; [s_4, s_2] = u_7^2,\; [s_5, s_2] = u_7^2,\; [s_4, s_3] = u_7, \\ & s_2^3 = s_4^2 s_5 u_5,\; s_3^3 = s_5^2 s_6 u_7^2,\; t_3^3 = u_5^2 u_7^2, u_5^3 = u_7^2, \\ & s_4^3 = s_6^2 s_7,\; s_5^3 = s_7^2,\; R(x) = 1,\; R(y) = 1 \;\rangle, \end{aligned}$$

where the relators $R(x)$ and $R(y)$ are given by equations (1) and (2) with $c = 7$.
The 3 isolated vertices $\langle 729, 49\rangle - \#2; 1 - \#1; 1 - \#2; 4$ of TKT E.6 and $\langle 729, 49\rangle - \#2; 1 - \#1; 1 - \#2; 5$, $\langle 729, 49\rangle - \#2; 1 - \#1; 1 - \#2; 6$ of TKT E.14 among the 6 descendants are Schur σ-groups.

FIGURE 7. Full normal lattice, including upper and lower central series, of a 3-group G with $G/G' \simeq (3,3)$, $|G| = 3^{11}$, $\mathrm{cl}(G) = 7$, $\mathrm{cc}(G) = 4$, $\mathrm{dl}(G) = 3$, satisfying $R(x) = 1$ and $R(y) = 1$ with relators $R(x), R(y)$ given by equations (1) and (2)



§ 3.5. The Schur σ-Groups

Concerning *even* branches of the trees $\mathcal{T}(\langle 243,6\rangle)$ and $\mathcal{T}(\langle 243,8\rangle)$, which are *admissible* as second 3-class groups $G_3^2(K)$ of quadratic number fields $K=\mathbb{Q}(\sqrt{D})$, we have:

Conjecture 3.5.
Let $n\ge 2$ be an integer. There exist exactly 6 pairwise non-isomorphic groups G of order 3^{3n+2}, class $2n+1$, coclass $n+1$, having fixed derived length 3, such that

(1) the factors of their upper central series are given by

$$\zeta_{j+1}(G)/\zeta_j(G)\simeq\begin{cases}(3,3) & \text{for } j=2n,\\ (3) & \text{for } 1\le j\le 2n-1,\\ (3,3^n) & \text{for } j=0,\end{cases}$$

(2) their second derived group $G''<\zeta_1(G)$ is central and cyclic of order 3^{n-1}.

Furthermore,

- they are Schur σ-groups with automorphism group $\mathrm{Aut}(G)$ of order $2\cdot 3^{4n+2}$,
- the factors of their lower central series are given by

$$\gamma_j(G)/\gamma_{j+1}(G)\simeq\begin{cases}(3,3) & \text{for odd } 1\le j\le 2n+1,\\ (3) & \text{for even } 2\le j\le 2n,\end{cases}$$

- their metabelianization G/G'' is of order 3^{2n+3}, class $2n+1$ and of fixed coclass 2,
- their biggest metabelian generalized predecessor, that is the $(2n-3)$rd generalized parent, is given by either $\langle 729,49\rangle$ or $\langle 729,54\rangle$.

§ 3.6. The Schur Cover Limit

Theorem 3.6. (Newman [2013])

The generalized periodic sequences of Schur σ-groups $(\overline{G^{(0)}_{2\ell+1,k+2}})_{\ell\geq 0}$, for $1 \leq k+2 \leq m = 3$, which form the unique Schur covers of vertices $G^{(0)}_{2\ell+1,k+2}$ of depth 1 on the TKT-pruned coclass tree $\mathcal{T}_2(G^{(0)}_0)$ with root $G^{(0)}_0 = \langle 243, 8\rangle$, resp. $\langle 243, 6\rangle$, are given as quotients $\overline{G^{(0)}_{2\ell+1,k+2}} = L_*/S_{k,c}$ of the infinite topological group

$$\begin{aligned} L_* \ = \ \langle\ & t, u, y, z, a \ \mid \\ & [[u,t],t] = 1,\ [[u,t],u] = 1, \\ & y^3 = 1,\ z^3 = 1,\ [y,z] = 1, \\ & [t,y] = 1,\ [u,y] = 1, \\ & [t,z] = 1,\ [u,z] = 1, \\ & [y,a] = 1,\ a^3[[t,a],t] = z, \\ & t^a = u,\ R = 1\ \rangle, \end{aligned}$$

with $R = u^a tuy[u,t]^{-1}$, resp. $R = u^a tuy$, by the closed subgroups $S_{k,c}$ generated by two elements,

$$y([t,a]^{(c-1)})^k[[t,a]^{(c-3)},[t,a]] \text{ and } z[t,a]^{(c-1)}$$

involving parameters $k \in \{-1, 0, 1\}$, resp. $k \in \{-1, 0\}$, and $c = 2\ell + 5 \geq 5$ odd. The group $L_*/S_{k,c}$ is of derived length 3, order $3^{(3c+1)/2}$, class c, and TKT
$\varkappa_1 = (2, 4, 3, 4)$, E.9, resp. $\varkappa_1 = (3, 4, 2, 3)$, E.14, for $k = -1$,
$\varkappa_2 = (2, 2, 3, 4)$, E.8, resp. $\varkappa_2 = (3, 2, 2, 3)$, E.6, for $k = 0$,
and $\varkappa_3 = (2, 3, 3, 4)$, E.9
[resp. $\varkappa_3 = (3, 3, 2, 3)$, H.4,
which must be excluded since it is not a Schur group],
for $k = 1$.

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