Der Wissenschaftsfonds.

p-Capitulation over Number Fields with p-Class Rank Two

Conference:	2nd International Conference on Groups and Algebras (ICGA 2016)
Place:	Youngor Central Hotel 63 Gongxiang Guanqian Street
Venue:	Suzhou, China
Date:	July 25 – 27, 2016
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A presentation within the frame of the international scientific research project

Towers of p-Class Fields over Algebraic Number Fields

supported by the Austrian Science Fund (FWF): P26008-N25

§ 0. Introduction

The aim of this lecture is
to communicate **the most up-to-date techniques**
for investigating the Hilbert p-class field tower

$$K \le \mathrm{F}_p^1 K \le \mathrm{F}_p^2 K \le \ldots \le \mathrm{F}_p^\infty K$$

of a finite algebraic number field $K|\mathbb{Q}$,
for an assigned prime number p.

The p-**capitulation type** $\varkappa(K)$ of K
is the first ingredient of this process.
Together with the **transfer target type** $\tau(K)$,
it forms the **Artin pattern** $\mathrm{AP}(K) = (\varkappa(K), \tau(K))$,
which determines the two-stage approximation

$$G := \mathrm{G}_p^2 K := \mathrm{Gal}\left(\mathrm{F}_p^2 K|K\right) \simeq H/H''$$

of the p-class tower group

$$H := \mathrm{G}_p^\infty K := \mathrm{Gal}\left(\mathrm{F}_p^\infty K|K\right).$$

To provide a concrete numerical **application**,
we have determined the metabelian group $\mathrm{G}_3^2 K$,
called the **second** 3**-class group**,
for the **34 631** real quadratic fields $K = \mathbb{Q}(\sqrt{d})$
with fundamental discriminants $0 < d < \mathbf{10^8}$ and
3-class group $\mathrm{Cl}_3 K$ of type $(3, 3)$.

0.1. Artin pattern: p-capitulation, transfer target

p . . . a prime number,
K . . . an algebraic number field,
$\mathrm{Cl}(K)$ the ideal class group of K,
$\mathrm{Cl}_pK := \mathrm{Syl}_p\mathrm{Cl}(K)$ the p-class group of K;

$L|K$ an *unramified* cyclic field extension of relative degree p,
$J_{L|K} : \mathrm{Cl}_pK \to \mathrm{Cl}_pL$ the p-class **transfer** homomorphism,
$\ker J_{L|K}$ the **p-capitulation kernel** (transfer kernel) of $L|K$,
Cl_pL the **transfer target** of $L|K$;

$E_p := \mathrm{Cl}_pK \bigotimes_{\mathbb{Z}_p} \mathbb{F}_p$ the p-elementary class group of K,
$\varrho := \dim_{\mathbb{F}_p} E_p$ the **p-class rank** of K,
U_K the group of units of K.

Proposition 0.1. (Bounds for the Order of $\ker J_{L|K}$)

$$1 \quad < \quad \ker J_{L|K} \quad \le \quad E_p \simeq \mathbb{F}_p^\varrho \qquad \text{and}$$

$$p \quad \le \quad \#\ker J_{L|K} = [L:K] \cdot (U_K : \mathrm{Norm}_{L|K} U_L) \quad \le \quad p^\varrho.$$

F_p^1K the Hilbert p-class field of K,
$\mathrm{Lyr}_1K := \{K \le L \le \mathrm{F}_p^1K \mid [L:K] = p\}$ . . .
. . . the first layer of unramified abelian extensions of K,
$n := \frac{p^\varrho-1}{p-1}$ the cardinality of $\mathrm{Lyr}_1K = \{L_1, \ldots, L_n\}$,
$\varkappa(K) := \left(\ker J_{L_i|K}\right)_{1\le i\le n}$ the **p-capitulation type** of K,
$\tau(K) := (\mathrm{Cl}_pL_i)_{1\le i\le n}$ the **transfer target type** of K.

In the sequel, we focus on $\varrho = 2$, whence $n = \frac{p^2-1}{p-1} = p + 1$.

0.2. Brief p-capitulation type and target type

In the case of $\mathrm{Cl}_p K \simeq (p,p)$,
which is relevant for our intended numerical application,
the components of the p-**capitulation type** of K,

$$\varkappa(K) := \left(\ker J_{L_i|K}\right)_{1\le i\le p+1},$$

are replaced by non-negative integers $j \in \{0,1,\ldots,p+1\}$.
For a **partial** capitulation, we put

$$\varkappa_i(K) = j :\Longleftrightarrow \ \ker J_{L_i|K} = \mathrm{Norm}_{L_j|K}\mathrm{Cl}_p L_j$$

with $1 \le j \le p+1$.
For a **total** capitulation, we put

$$\varkappa_i(K) = 0 :\Longleftrightarrow \ \ker J_{L_i|K} = \mathrm{Cl}_p K.$$

(For $\mathrm{Cl}_p K \simeq (p^u,p)$ or (p^u,p^v), see the Appendix II.)

The components of the **transfer target type** of K,

$$\tau(K) := (\mathrm{Cl}_p L_i)_{1\le i\le p+1},$$

are replaced by the logarithmic form of abelian type invariants,
$\mathrm{Cl}_p L_i \simeq C(p^{e_1})^{m_1} \times \ldots \times C(p^{e_s})^{m_s} \ \Longleftrightarrow : \tau_i(K) = (e_1^{m_1},\ldots,e_s^{m_s})$.

Example:
If $p=3$, then $n=4$, and
$\mathrm{Cl}_3 L_1 \simeq C(9)^2$, $\mathrm{Cl}_3 L_i \simeq C(3)^2$ for $2 \le i \le 4$
$\Longleftrightarrow : \tau(K) = \left(2^2, (1^2)^3\right)$.

0.3. Main theorem on class, coclass, defect of $\mathrm{G}_3^2 K$

Theorem 0.1. (The Second 3-Class Group)
Let $p = 3$, then the transfer target type $\tau(K) := (\mathrm{Cl}_3 L_i)_{1 \le i \le 4}$
of a number field K with 3-class group $\mathrm{Cl}_3 K \simeq (3,3)$
determines the nilpotency class $c = \mathrm{cl}(G)$,
the coclass $r = \mathrm{cc}(G)$, and the defect $k = k(G)$
of the second 3-class group $G = \mathrm{G}_3^2 K$ as follows:

$$\#\mathrm{Cl}_3 L_1 = 3^{c-k},$$
$$\#\mathrm{Cl}_3 L_2 = 3^{r+1},$$
$$\#\mathrm{Cl}_3 L_i = 3^2 \text{ for } 3 \le i \le 4, \text{ if } r = 1,$$
$$\#\mathrm{Cl}_3 L_i = 3^3 \text{ for } 3 \le i \le 4, \text{ if } r \ge 2.$$

(Proof: [1] D. C. Mayer, IJNT **8** (2012), no. 2, 471–505.)

Corollary 0.1. (The Hilbert 3-Class Field)
The 3-class group of the Hilbert 3-class field $\mathrm{F}_3^1 K$
of a number field K with 3-class group $\mathrm{Cl}_3 K \simeq (3,3)$
admits the separation of the nilpotency class $c = \mathrm{cl}(G)$
and the defect $k = k(G)$ with $c - k \ge r + 1$:

$$\#\mathrm{Cl}_3 \mathrm{F}_3^1 K = 3^{c+r-2}.$$

Example:
$\tau(K) = \left(2^2, (1^2)^3\right)$ and $\mathrm{Cl}_3 \mathrm{F}_3^1 K \simeq 2^2$ together imply
$c - k = 4$, $r + 1 = 2$, $c + r - 2 = 4$, and thus
$r = 1$, $c = 5$ and $k = 1$.

§§ 1–4. Preliminary Observations

The large-scale computation
of the second 3-class group $\mathrm{G}_3^2 K$ for the
34 631 real quadratic fields $K = \mathbb{Q}(\sqrt{d})$
with fundamental discriminants $0 < d < \mathbf{10^8}$ and
3-class group $\mathrm{Cl}_3 K$ of type $(3,3)$
would have been impossible prior to the release of
MAGMA Version V2.21-8 in November 2015.

Even more striking, the large-scale computation
of the second 3-class group $\mathrm{G}_3^2 K$ for the
24 476 imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$
with fundamental discriminants $-\mathbf{10^7} < d < 0$ and
3-class group $\mathrm{Cl}_3 K$ of type $(3,3)$ in Appendix I
was definitely outside of the scope before
MAGMA Version V2.22-1, released June 2016.

The Computational Algebra Group,
MAGMA Computational Algebra System, Version 2.22-3,
Univ. Sydney, 2016,
(`http://magma.maths.usyd.edu.au`).

W. Bosma, J. Cannon, and C. Playoust,
The Magma algebra system. I. The user language,
J. Symbolic Comput. **24** (1997), 235–265.

C. Fieker, *Computing class fields via the Artin map*,
Math. Comp. **70** (2001), no. 235, 1293–1303.

§ 1. Statistics for $\mathrm{cc}(\mathrm{G}_3^2 K) = 1$

Table 1. 3-Capitulation Types a

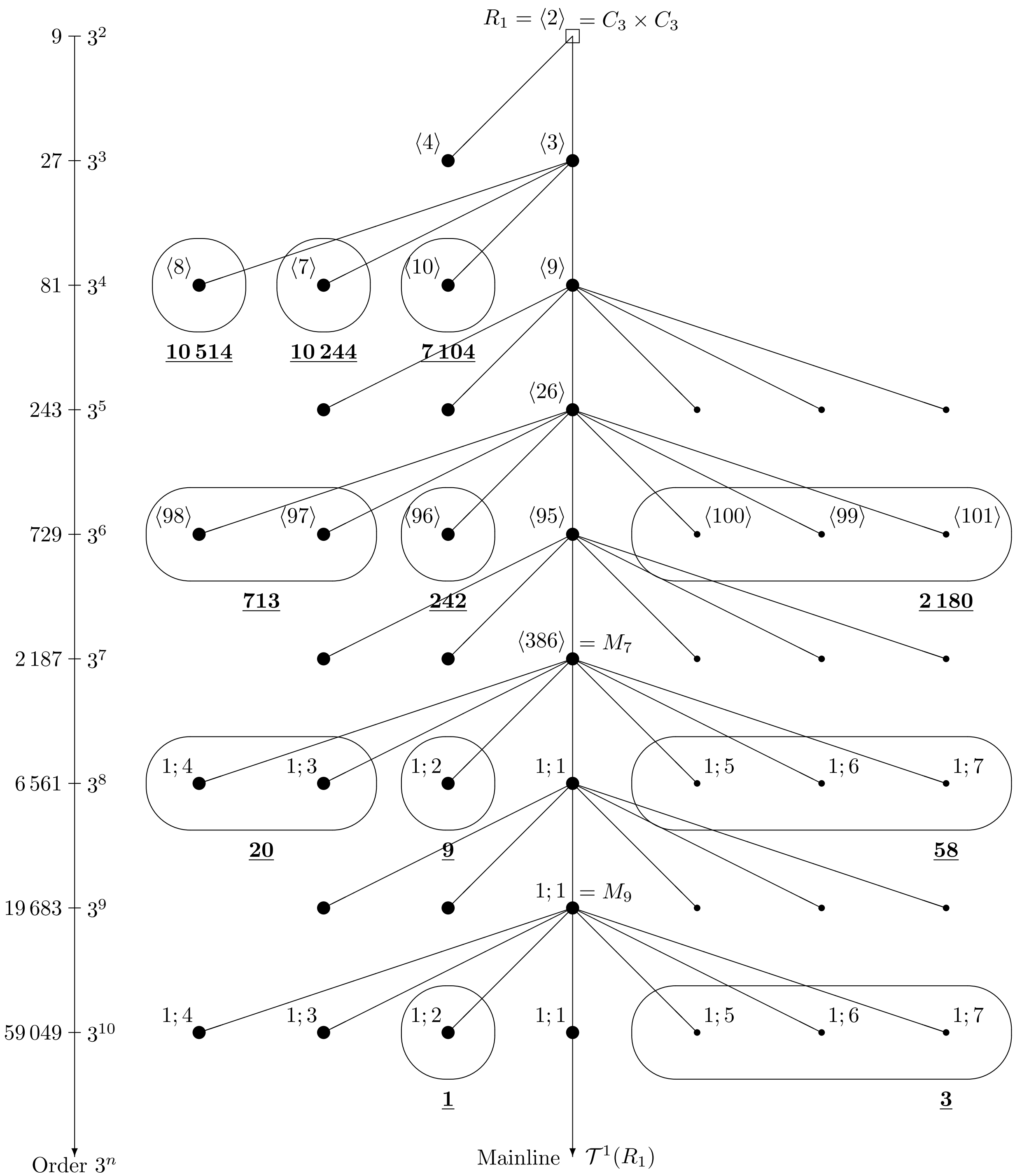
Type	Artin Pattern		Freq.	$\mathrm{G}_3^2 K$ [2, 3]	#
	$\varkappa$	τ			
a.2	(1000)	$21, (1^2)^3$	7 104	$\langle 3^4, 10\rangle$	1
a.3	(2000)	$21, (1^2)^3$	10 514	$\langle 3^4, 8\rangle$	1
a.3∗	(2000)	$1^3, (1^2)^3$	10 244	$\langle 3^4, 7\rangle$	1
a.1	(0000)	$2^2, (1^2)^3$	2 180	$\langle 3^6, 99\ldots 101\rangle$	3
a.2↑	(1000)	$32, (1^2)^3$	242	$\langle 3^6, 96\rangle$	1
a.3↑	(2000)	$32, (1^2)^3$	713	$\langle 3^6, 97\vert 98\rangle$	2
a.1↑	(0000)	$3^2, (1^2)^3$	58	$M_7 - \#1; 5\ldots 7$	3
$\mathrm{a}.2\uparrow^2$	(1000)	$43, (1^2)^3$	9	$M_7 - \#1; 2$	1
$\mathrm{a}.3\uparrow^2$	(2000)	$43, (1^2)^3$	20	$M_7 - \#1; 3\vert 4$	2
$\mathrm{a}.1\uparrow^2$	(0000)	$4^2, (1^2)^3$	3	$M_9 - \#1; 5\ldots 7$	3
$\mathrm{a}.2\uparrow^3$	(1000)	$54, (1^2)^3$	1	$M_9 - \#1; 2$	1
		Total:	31 088	89.77% of 34 631	

In Table 1, $M_7 := \langle 3^7, 386\rangle$ and $M_9 := M_7 - \#1; 1 - \#1; 1$.

Theorem 1.1. (Two-Stage Towers)
The 3-class field tower of a real quadratic field K with 3-capitulation $\varkappa(K)$ of type a.1 or a.2 or a.3 (either in the ground state or in any excited state) possesses the exact length $\ell_3 K = 2$.
The 3-class tower group $H = \mathrm{G}_3^\infty K$ with $H'' = 1$ is isomorphic to its metabelianization $H/H'' \simeq \mathrm{G}_3^2 K$.

(Proof: [4] D. C. Mayer, arXiv: 1601.00179v1 [math.NT].)

FIGURE 1. Population of the unique coclass-1 tree $\mathcal{T}^1(C_3 \times C_3)$



Type	a.3	a.3	a.2	a.1	a.1	a.1	a.1
$\varkappa =$	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

Table 1 and Figure 1 impressively demonstrate the wealth of realizations of finite 3-groups G of coclass $\mathrm{cc}(G) = 1$, also called 3-groups of **maximal class**, by second 3-class groups $\mathrm{G}_3^2 K$ of **real quadratic fields** K.

(Note that for imaginary quadratic fields, whose 3-capitulation can only be partial by Proposition 0.1, the tree $\mathcal{T}^1(C_3 \times C_3)$ is entirely forbidden.)

With nearly 90%, precisely 89.77%, of 34 631 fields this contribution of 31 088 fields is definitely **dominating**.

The large-scale **separation of the types a.2 and a.3** became feasible for the first time **by our new algorithm**. It shows that the *ground state* of type a.3 alone covers 30.36% (nearly a third) with its 10 514 hits.

In view of Theorem 1.1, the computed distribution proves that 90% of the **real quadratic fields** are happy with a **tower of only two stages**, though the *order* of $\mathrm{G}_3^\infty K$ is *unbounded*, due to the existence of infinitely many *excited states* $\uparrow, \uparrow^2, \ldots$

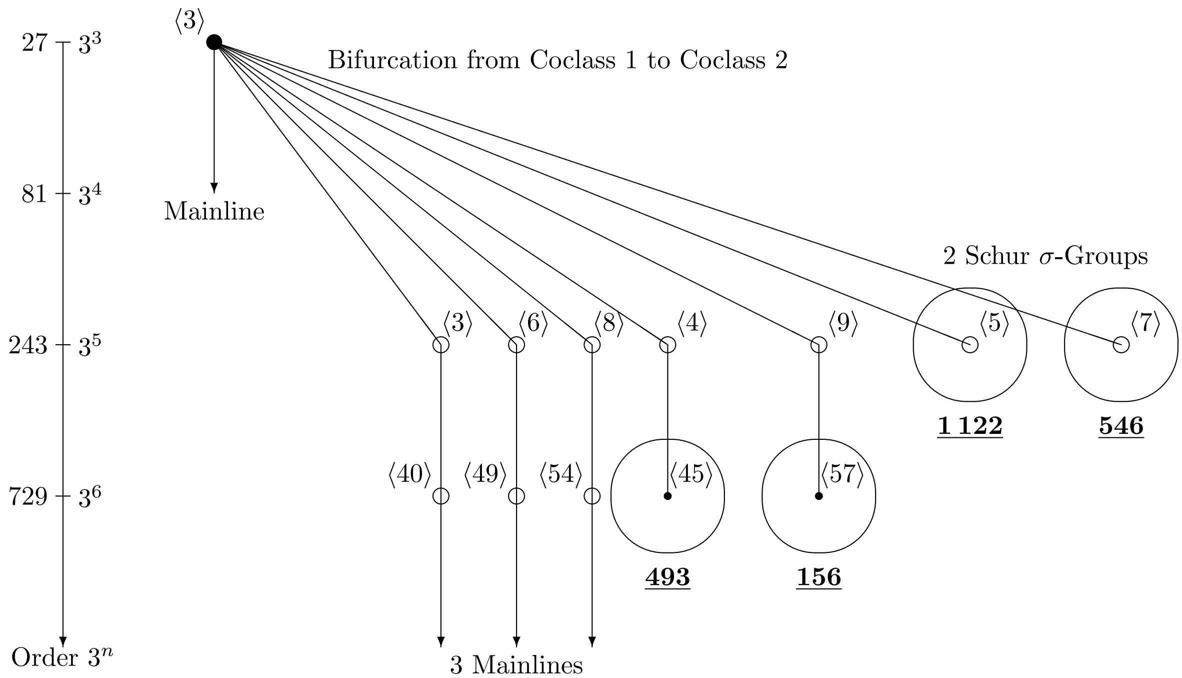
§ 2. Statistics for $\mathrm{cc}(\mathrm{G}_3^2 K) = 2$

Table 2. 3-Capitulation Types c, D, E, G and H

Type	Artin Pattern		Freq.	$\mathrm{G}_3^2 K$ [2, 3]	#
	$\varkappa$	τ			
D.5∗	(4224)	$1^3, 21, 1^3, 21$	546	$\langle 3^5, 7\rangle$	1
D.10∗	(2241)	$21, 21, 1^3, 21$	1 122	$\langle 3^5, 5\rangle$	1
G.19∗	(2143)	$(21)^4$	156	$\langle 3^6, 57\rangle$	1
H.4∗	(4443)	$(1^3)^2, 21, 1^3$	493	$\langle 3^6, 45\rangle$	1
c.18	(0313)	$2^2, 21, 1^3, 21$	347	$\langle 3^6, 49\rangle$	1
E.6	(1313)	$32, 21, 1^3, 21$	40	$\langle 3^7, 288\rangle$	1
E.14	(2313)	$32, 21, 1^3, 21$	63	$\langle 3^7, 289\vert 290\rangle$	2
H.4	(3313)	$32, 21, 1^3, 21$	37	$O_{7,3\vert 4} - \#1; 2$	2
c.21	(0231)	$2^2, (21)^3$	358	$\langle 3^6, 54\rangle$	1
E.8	(1231)	$32, (21)^3$	30	$\langle 3^7, 304\rangle$	1
E.9	(2231)	$32, (21)^3$	83	$\langle 3^7, 302\vert 306\rangle$	2
G.16	(4231)	$32, (21)^3$	27	$S_{7,1\vert 5} - \#1; 4$	2
c.18↑	(0313)	$3^2, 21, 1^3, 21$	8	Q_8	1
E.6↑	(1313)	$43, 21, 1^3, 21$	1	$Q_8 - \#1; 4$	1
E.14↑	(2313)	$43, 21, 1^3, 21$	1	$Q_8 - \#1; 5\vert 6$	2
c.21↑	(0231)	$3^2, (21)^3$	12	U_8	1
E.8↑	(1231)	$43, (21)^3$	2	$U_8 - \#1; 2$	1
E.9↑	(2231)	$43, (21)^3$	1	$U_8 - \#1; 4\vert 6$	2
G.16↑	(4231)	$43, (21)^3$	1	$S_{9,3\vert 5} - \#1; 2$	2
		Total:	3 328	9.61% of 34 631	

In Table 2, $Q_8 := \langle 3^7, 285\rangle - \#1; 1$, $U_8 := \langle 3^7, 303\rangle - \#1; 1$, $O_{7,3|4} := \langle 3^7, 286|287\rangle$, $S_{7,1|5} := \langle 3^7, 301|305\rangle$, and $S_{9,3|5} := U_8 - \#1; 3|5$.

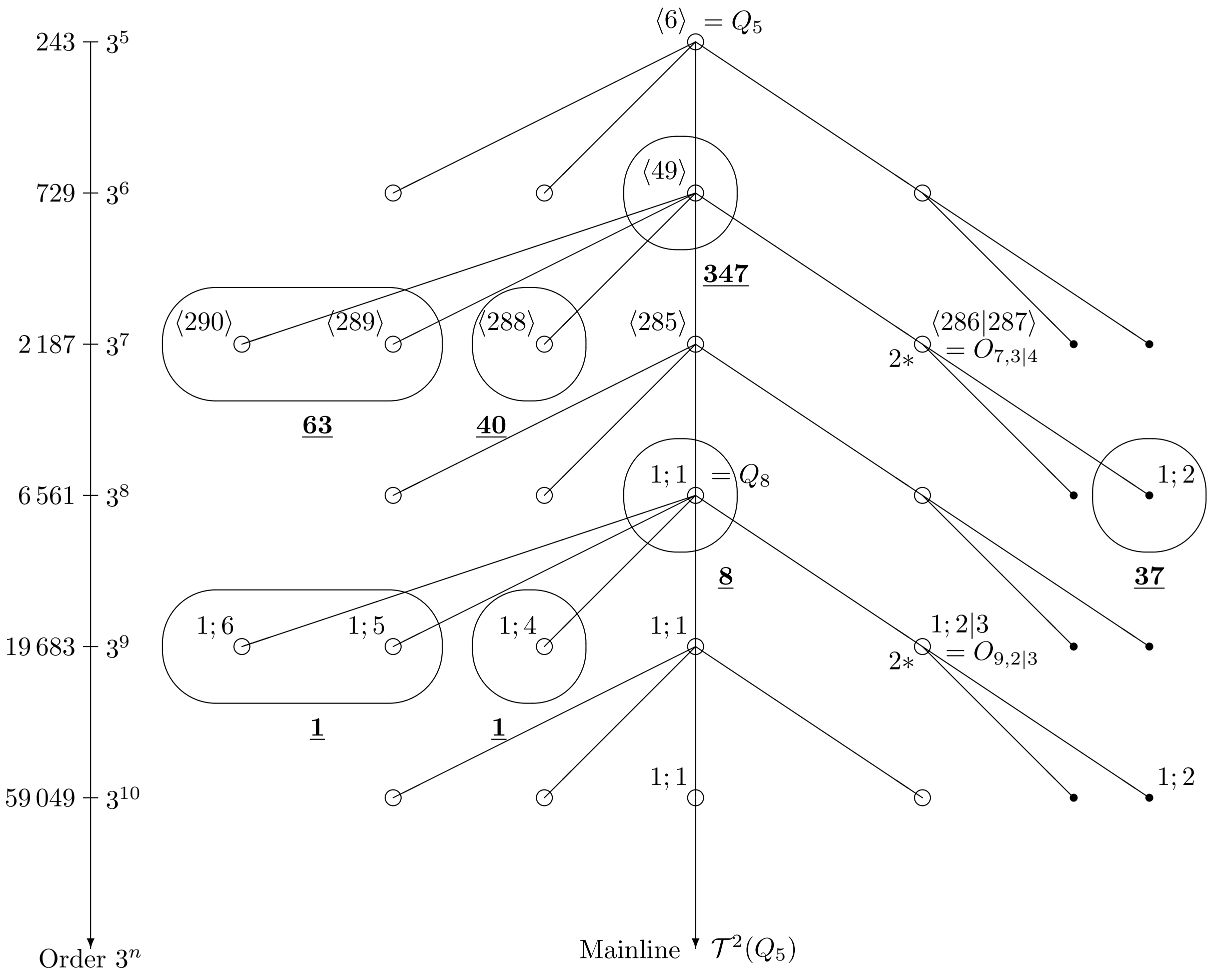
FIGURE 2. Population of the sporadic coclass-2 graph $\mathcal{G}_0(3,2)$



Type	H.4∗	G.19∗	D.10∗	D.5∗
$\varkappa =$	(3313)	(2143)	(2241)	(4224)

In contrast to the coclass-1 graph $\mathcal{G}(3,1)$, which consists of the single coclass tree $\mathcal{T}^1(C_3 \times C_3)$, the coclass-2 graph $\mathcal{G}(3,2)$ is a **forest** which is constituted by a **finite sporadic part** $\mathcal{G}_0(3,2)$ and **two admissible coclass trees** $\mathcal{T}^2(Q_5)$ and $\mathcal{T}^2(U_5)$. (There is yet another coclass tree $\mathcal{T}^2(B_5)$ with root $\langle 3^5, 3\rangle$ which is entirely forbidden, even for real quadratic fields.) Consequently, we must split the graphical representation of the distribution in Table 2 into three Figures 2, 3, and 4.

FIGURE 3. Population of the first admissible coclass-2 tree $\mathcal{T}^2(Q_5)$

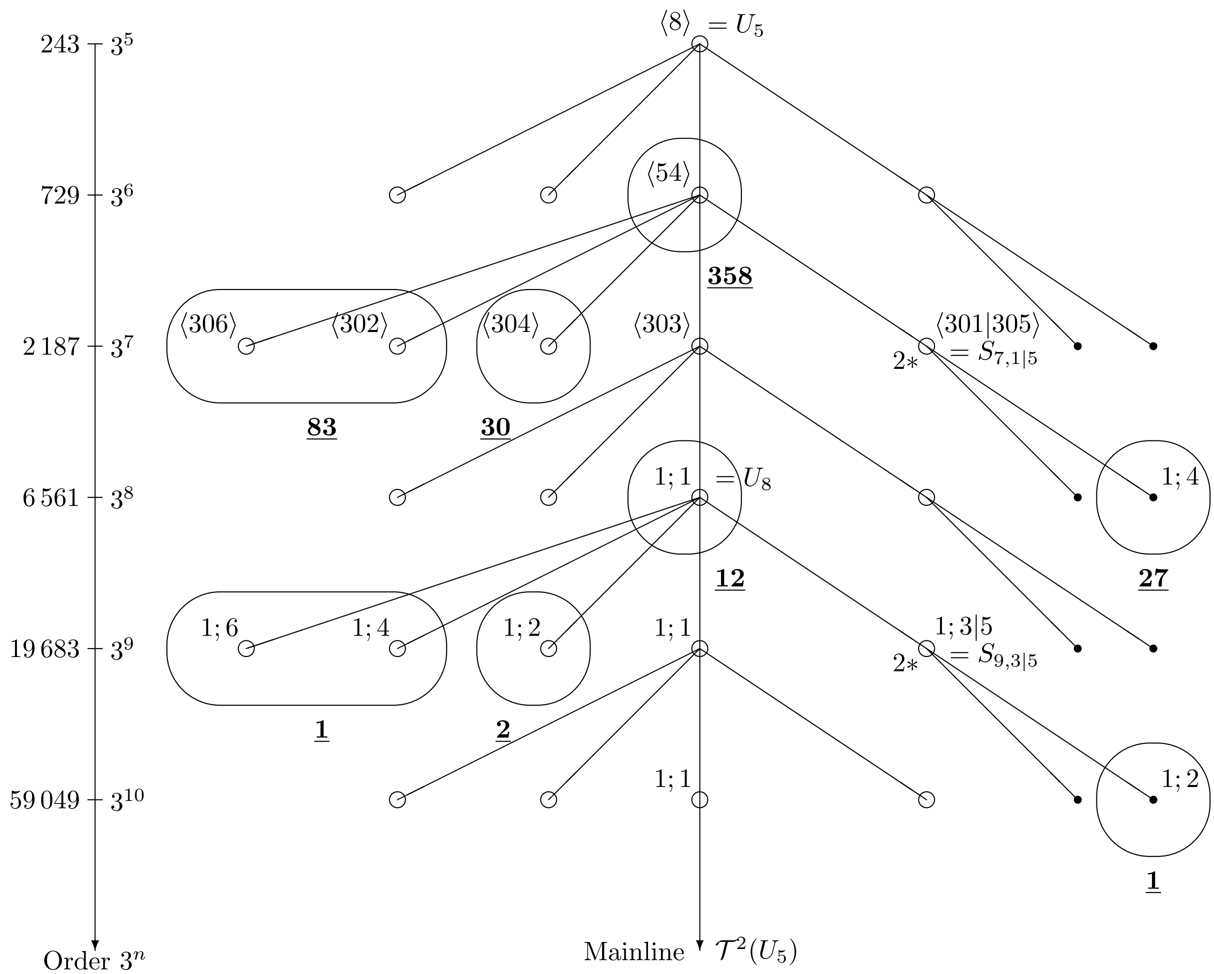


Type	E.14	E.14	E.6	c.18	H.4
$\varkappa =$	(2313)	(2313)	(1313)	(0313)	(3313)

The large-scale **separation of the types E.6 and E.14** became feasible for the first time **by our new algorithm**.

For identifying the groups $\mathrm{G}_3^2 K$ of type H.4, it was **not necessary** to compute Hilbert 3-class fields $\mathrm{F}_3^1 K$ of absolute degree 18.

FIGURE 4. Population of the second admissible coclass-2 tree $\mathcal{T}^2(U_5)$



Type	E.9	E.9	E.8	c.21		G.16
$\varkappa =$	(2231)	(2231)	(1231)	(0231)		(4231)

The large-scale **separation of the types E.8 and E.9** became feasible for the first time **by our new algorithm**.

For identifying the groups $\mathrm{G}_3^2 K$ of type G.16, it was **not necessary** to compute Hilbert 3-class fields $\mathrm{F}_3^1 K$ of absolute degree 18.

Theorem 2.1. (Three-Stage Towers)
The 3-class field tower of a real quadratic field K
with 3-capitulation $\varkappa(K)$ of type c.18 or c.21
(either in the ground state or in any excited state)
possesses the exact length $\ell_3 K = 3$.
The 3-class tower group $H = \mathrm{G}_3^\infty K$ with $H'' > 1$
is a proper extension of its metabelianization $H/H'' \simeq \mathrm{G}_3^2 K$.

(Proof: [5] D. C. Mayer, TMMP **64** (2015), 21 – 57.)

Theorem 2.2. (Two- or Three-Stage Towers)
The 3-class field tower of a real quadratic field K
with 3-capitulation $\varkappa(K)$ of type E.6 or E.8 or E.9 or E.14
(either in the ground state or in any excited state)
possesses a length bounded by $2 \le \ell_3 K \le 3$.

(Proof:
[6] D. C. Mayer, APM **5** (2015), no. 4, 162–195,
[7] D. C. Mayer, JAMP **3** (2015), no. 7, 746–756, and
[8] D. C. Mayer, APM **5** (2015) no. 5, 286–313.)

Additionally, we mention that the 3-class tower groups $\mathrm{G}_3^\infty K$ of real quadratic fields K with type D.5$*$ or D.10$*$ have fixed order 3^5 and derived length 2, and that the smallest possible 3-class tower groups $\mathrm{G}_3^\infty K$ of real quadratic fields K with type G.19$*$ or H.4$*$ have order 3^7 and derived length 3.

(Proof: [4] D. C. Mayer, arXiv: 1601.00179v1 [math.NT].)

§ 3. Statistics for $\mathrm{cc}(\mathrm{G}_3^2 K) = 3$

Table 3. 3-Capitulation Types b and d

Type	Artin Pattern		Freq.	$\mathrm{G}_3^2 K$ [2, 3]	#
	$\varkappa$	τ			
b.10	(0043)	$(2^2)^2, (1^3)^2$	95	$P_7 - \#1; 21 \ldots 26$	6
d.19	(4043)	$32, 2^2, (1^3)^2$	49	$P_7 - \#1; 4\|5$	2
d.23	(1043)	$32, 2^2, (1^3)^2$	16	$P_7 - \#1; 6$	1
d.25	(2043)	$32, 2^2, (1^3)^2$	22	$P_7 - \#1; 7\|8$	2
b.10↑	(0043)	$3^2, 2^2, (1^3)^2$	6	$P_9 - \#1; 21 \ldots 29$	9
d.19↑	(4043)	$43, 2^2, (1^3)^2$	1	$P_9 - \#1; 2\|3$	2
d.23↑	(1043)	$43, 2^2, (1^3)^2$	1	$P_9 - \#1; 4$	1
		Total:	190	0.55% of 34 631	

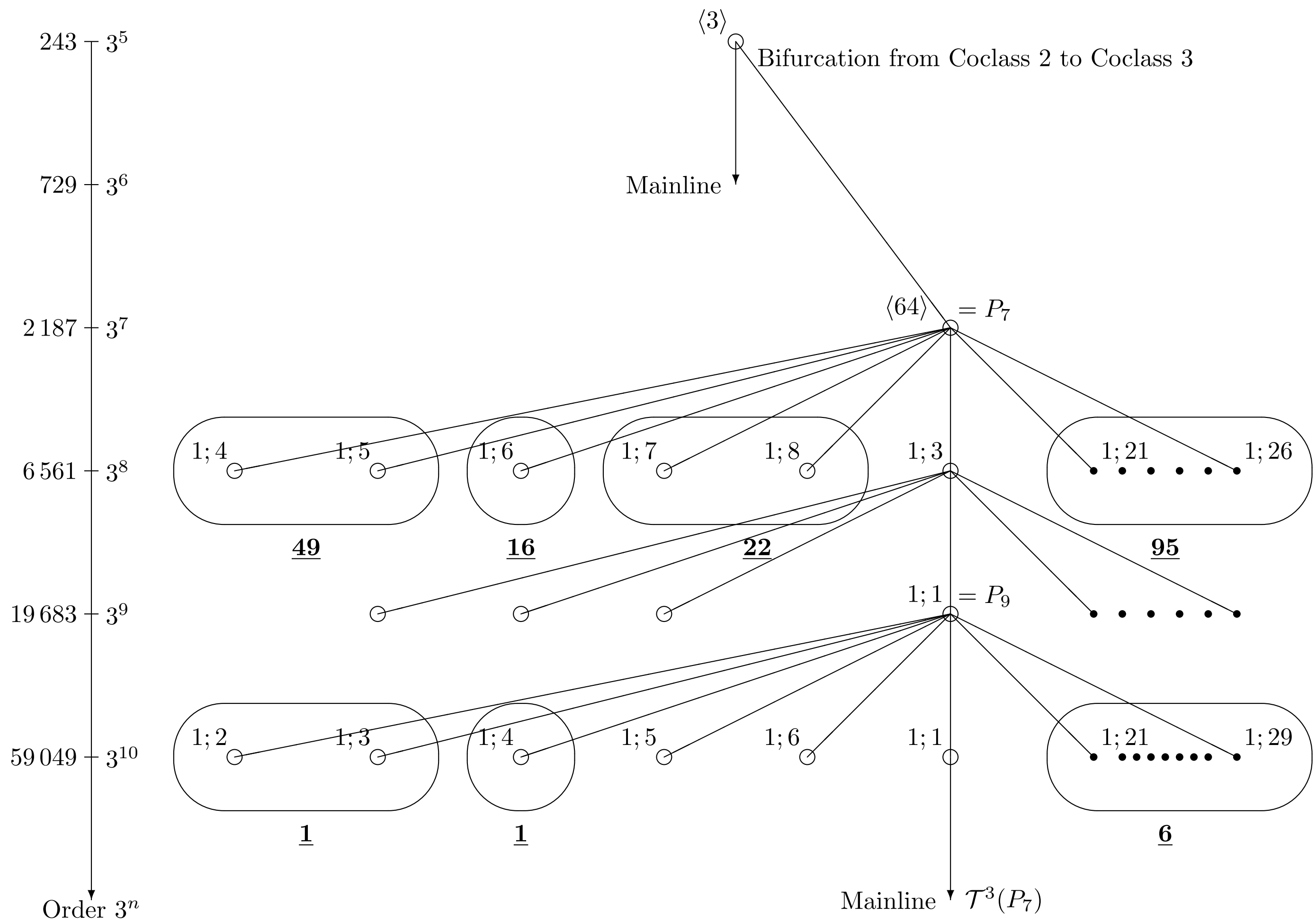
In Table 3, $P_7 := \langle 3^7, 64\rangle$ and $P_9 := P_7 - \#1; 3 - \#1; 1$.

The coclass-3 graph $\mathcal{G}(3,3)$ is a **forest**, constituted by
a forbidden finite sporadic part $\mathcal{G}_0(3,3)$,
three coclass trees which are forbidden for any quadratic field,
and a **single admissible coclass tree** $\mathcal{T}^3(P_7)$.

The graphical representation
of the distribution in Table 3 is drawn in Figure 5.

The smallest possible 3-class tower groups $\mathrm{G}_3^\infty K$
of real quadratic fields K with type b.10, d.19, d.23 or d.25
have order 3^9 and derived length 3.

FIGURE 5. Population of the admissible coclass-3 tree $\mathcal{T}^3(P_7)$



Type	d.19	d.19	d.23	d.25	d.25	b.10	b.10	b.10
$\varkappa =$	(4043)	(4043)	(1043)	(2043)	(2043)	(0043)	(0043)	(0043)

In the restricted range $0 < d < 10^7$, investigated in 2010, the occurrence of type b.10, d.19, d.23 was still highly **exotic**, with 8, 1, 1 hits by $\mathrm{G}_3^2 K$ for $K = \mathbb{Q}(\sqrt{d})$, respectively.
Now we have a rather representative collection of examples for the extended range $0 < d < 10^8$.
The **terminal variant** of type d.25 appears here for the first time, and even some *excited states* show up.

§ 4. Statistics for $\mathrm{cc}(\mathrm{G}_3^2 K) = 4$

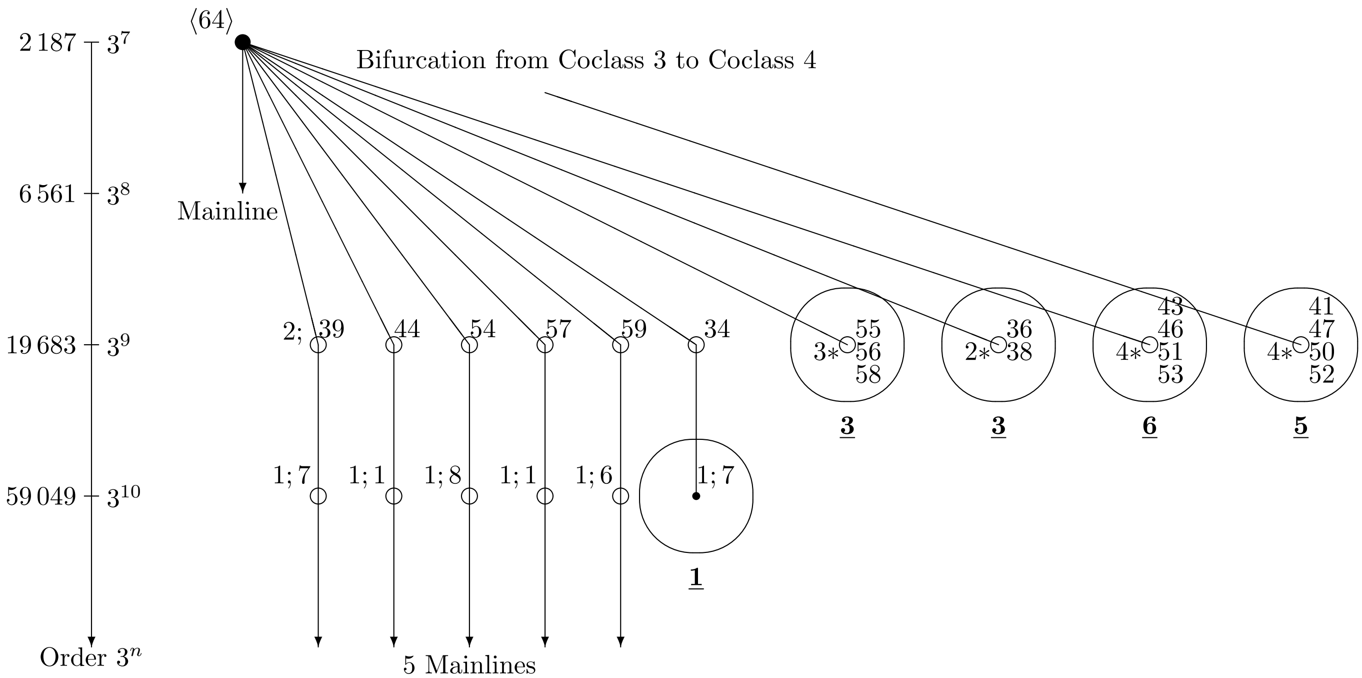
Table 4. 3-Capitulation Types d, F and H

Type	Artin Pattern		Freq.	$\mathrm{G}_3^2 K$ [2, 3]	#
	$\varkappa$	τ			
F.7∗	(3443)	$(32)^2, (1^3)^2$	3	$S_{9,55\|56\|58}$	3
F.11∗	(1143)	$(32)^2, (1^3)^2$	3	$S_{9,36\|38}$	2
F.12∗	(1343)	$(32)^2, (1^3)^2$	6	$S_{9,43\|46\|51\|53}$	4
F.13∗	(3143)	$(32)^2, (1^3)^2$	5	$S_{9,41\|47\|50\|52}$	4
H.4i∗	(4443)	$(32)^2, (1^3)^2$	1	$S_{9,34} - \#1; 7$	1
d.25m	(0143)	$3^2, 32, (1^3)^2$	4	$S_{10,57\|59}$	2
F.7	(3443)	$43, 32, (1^3)^2$	1	$S_{10,39\|44} - \#1; 5\|6$	4
F.12	(1343)	$43, 32, (1^3)^2$	1	$S_{10,39} - \#1; 2\|9$, $S_{10,44} - \#1; 3\|8$,	
				$S_{10,54} - \#1; 2\|4\|6\|8$	8
F.13	(3143)	$43, 32, (1^3)^2$	1	$S_{10,39} - \#1; 3\|8$, $S_{10,44} - \#1; 2\|9$,	
				$S_{10,57} - \#1; 2\|4$, $S_{10,59} - \#1; 3\|4$	8
		Total:	25	0.07% of 34 631	

In Table 4, $S_{9,j} := P_7 - \#2; j$, where $P_7 := \langle 3^7, 64 \rangle$, and $S_{10,39} := S_{9,39} - \#1; 7$, $S_{10,44} := S_{9,44} - \#1; 1$, $S_{10,54} := S_{9,54} - \#1; 8$, $S_{10,57} := S_{9,57} - \#1; 1$, $S_{10,59} := S_{9,59} - \#1; 6$.

The coclass-4 graph $\mathcal{G}(3,4)$ is a **forest**, constituted by
a **finite sporadic part** $\mathcal{G}_0(3,4)$
and **five admissible coclass trees** $\mathcal{T}^4(S_{9,j})$
with $j \in \{39, 44, 54, 57, 59\}$.
(There is yet another coclass tree $\mathcal{T}^4(S_{9,33})$
which is entirely forbidden, even for real quadratic fields.)
We restrict ourselves to the graphical representation
of the sporadic distribution in Table 4 by Figure 6.

FIGURE 6. Population of the sporadic coclass-4 graph $\mathcal{G}_0(3,4)$



Type	H.4i∗	F.7∗	F.11∗	F.12∗	F.13∗
$\varkappa =$	(4443)	(3443)	(1143)	(1343)	(3143)

The smallest possible 3-class tower groups $\mathrm{G}_3^\infty K$ of real quadratic fields K with type F.7, F.11, F.12 or F.13 have order 3^{10} and derived length 3.

The symbol d.25m in Table 4 indicates the **mainline variant** of type d.25 which differs essentially from the terminal variant of type d.25 in Table 3.

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APPENDIX I

Numerical Application to Imaginary Quadratic Fields

§ 1. Our Second Large-Scale Computation in June 2016 with MAGMA V2.22-1

To enable an intriguing **comparison**,
we have determined the metabelian group $\mathrm{G}_3^2 K$,
called the **second 3-class group**,
for the **24 476** imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$
with fundamental discriminants $-\mathbf{10^7} < d < 0$ and
3-class group $\mathrm{Cl}_3 K$ of type $(3, 3)$.

The most **striking difference**
to real quadratic fields is
the restriction of the distribution
of second 3-class groups $\mathrm{G}_3^2 K$ to
coclass graphs $\mathcal{G}(3, r)$ with **even coclass** $r \geq 2$.

The lack of odd coclass is compensated
by the population of even coclass
with decisively **higher density**
and by the occurrence of the elevated coclass $r = 6$.

Thus we have three Tables 5, 6, and 7
for $r = 2$, $r = 4$, and $r = 6$.

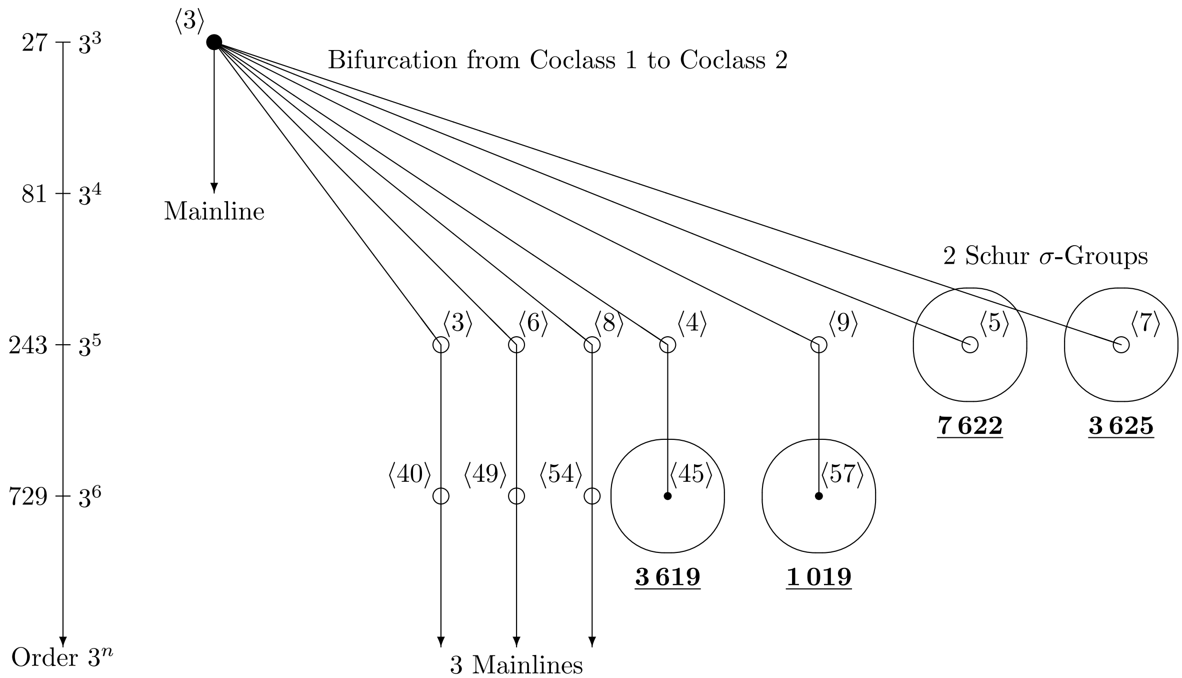
§ 2. Statistics for $\mathrm{cc}(\mathrm{G}_3^2 K) = 2$

Table 5. 3-Capitulation Types D, E, G and H

Type	Artin Pattern		Freq.	$\mathrm{G}_3^2 K$	#
	$\varkappa$	τ			
D.5∗	(4224)	$1^3, 21, 1^3, 21$	3 625	$\langle 3^5, 7\rangle$	1
D.10∗	(2241)	$21, 21, 1^3, 21$	7 622	$\langle 3^5, 5\rangle$	1
G.19∗	(2143)	$(21)^4$	1 019	$\langle 3^6, 57\rangle$	1
H.4∗	(4443)	$(1^3)^2, 21, 1^3$	3 619	$\langle 3^6, 45\rangle$	1
E.6	(1313)	$32, 21, 1^3, 21$	760	$\langle 3^7, 288\rangle$	1
E.14	(2313)	$32, 21, 1^3, 21$	1 572	$\langle 3^7, 289\vert 290\rangle$	2
H.4	(3313)	$32, 21, 1^3, 21$	781	$O_{7,3\vert 4} - \#1; 2$	2
E.8	(1231)	$32, (21)^3$	798	$\langle 3^7, 304\rangle$	1
E.9	(2231)	$32, (21)^3$	1 583	$\langle 3^7, 302\vert 306\rangle$	2
G.16	(4231)	$32, (21)^3$	809	$S_{7,1\vert 5} - \#1; 4$	2
E.6↑	(1313)	$43, 21, 1^3, 21$	88	$Q_8 - \#1; 4$	1
E.14↑	(2313)	$43, 21, 1^3, 21$	164	$Q_8 - \#1; 5\vert 6$	2
H.4↑	(3313)	$43, 21, 1^3, 21$	85	$O_{9,2\vert 3} - \#1; 2$	2
E.8↑	(1231)	$43, (21)^3$	82	$U_8 - \#1; 2$	1
E.9↑	(2231)	$43, (21)^3$	147	$U_8 - \#1; 4\vert 6$	2
G.16↑	(4231)	$43, (21)^3$	100	$S_{9,3\vert 5} - \#1; 2$	2
		Total:	22 900	93.56% of 24 476	

In Table 5, $Q_8 := \langle 3^7, 285\rangle - \#1; 1$, $U_8 := \langle 3^7, 303\rangle - \#1; 1$, $O_{7,3|4} := \langle 3^7, 286|287\rangle$, $O_{9,2|3} := Q_8 - \#1; 2|3$, $S_{7,1|5} := \langle 3^7, 301|305\rangle$, and $S_{9,3|5} := U_8 - \#1; 3|5$.

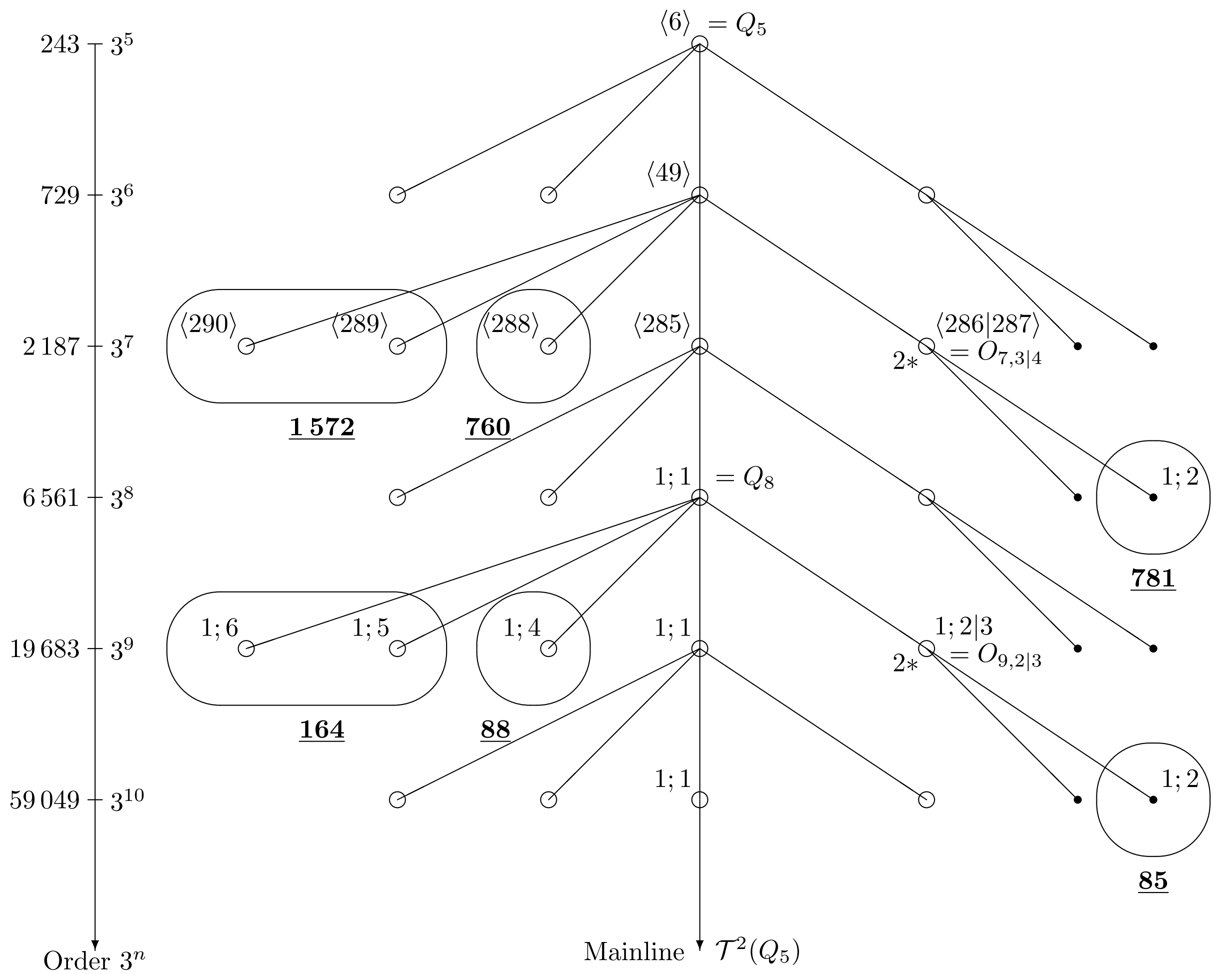
FIGURE 7. Population of the sporadic coclass-2 graph $\mathcal{G}_0(3,2)$



Type	H.4∗	G.19∗	D.10∗	D.5∗
$\varkappa =$	(3313)	(2143)	(2241)	(4224)

The definite **high-champ** of the distribution is type D.10∗ which alone covers 33.28% (nearly a third) with its 7 622 hits.

FIGURE 8. Population of the first admissible coclass-2 tree $\mathcal{T}^2(Q_5)$

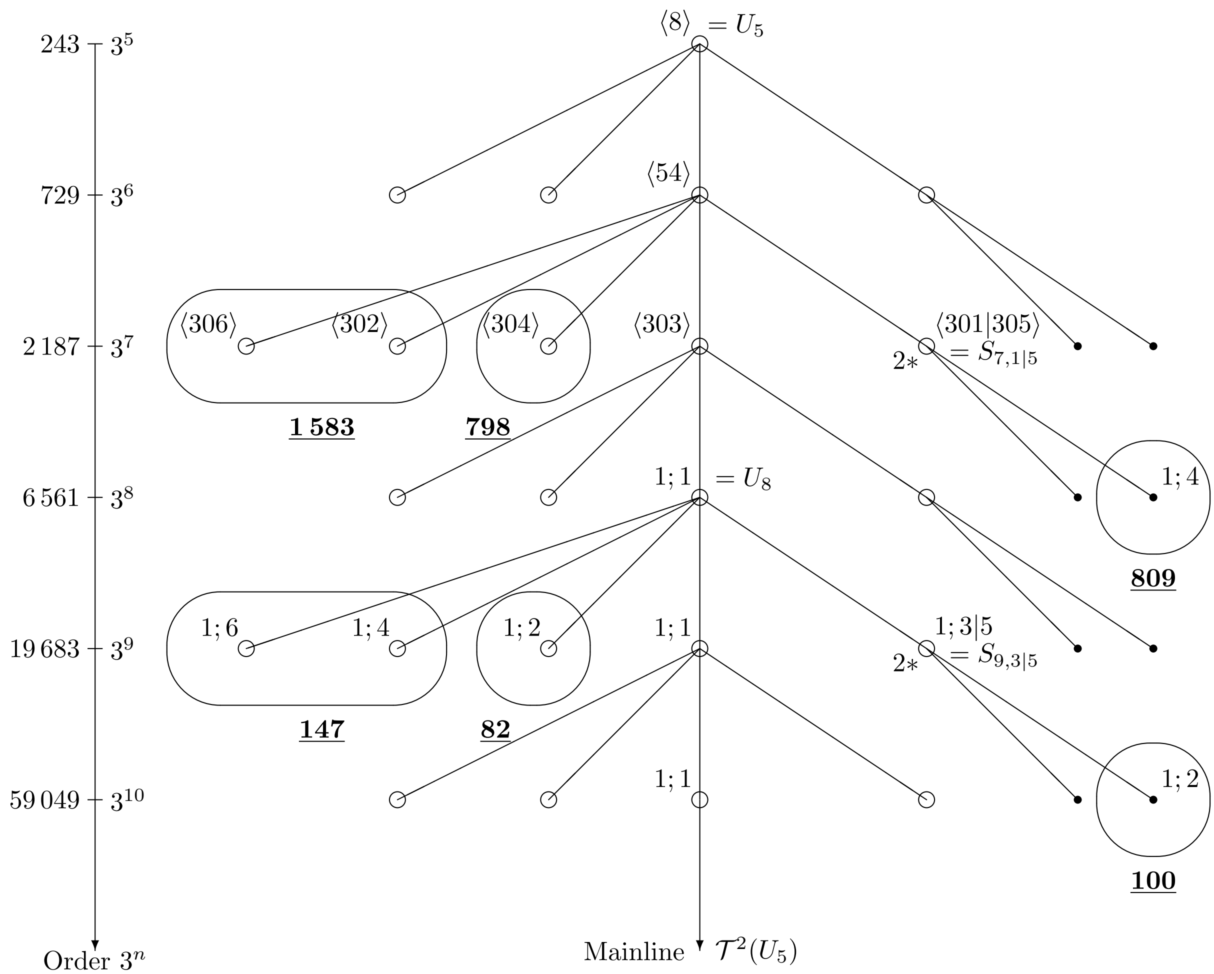


Type	E.14	E.14	E.6	c.18	H.4
$\varkappa =$	(2313)	(2313)	(1313)	(0313)	(3313)

The large-scale **separation of the types E.6 and E.14** became feasible for the first time **by our new algorithm**.

For identifying the groups $\mathrm{G}_3^2 K$ of type H.4, it was **not necessary** to compute Hilbert 3-class fields $\mathrm{F}_3^1 K$ of absolute degree 18.

FIGURE 9. Population of the second admissible coclass-2 tree $\mathcal{T}^2(U_5)$



Type	E.9	E.9	E.8	c.21			G.16
$\varkappa =$	(2231)	(2231)	(1231)	(0231)			(4231)

The large-scale **separation of the types E.8 and E.9** became feasible for the first time **by our new algorithm**.

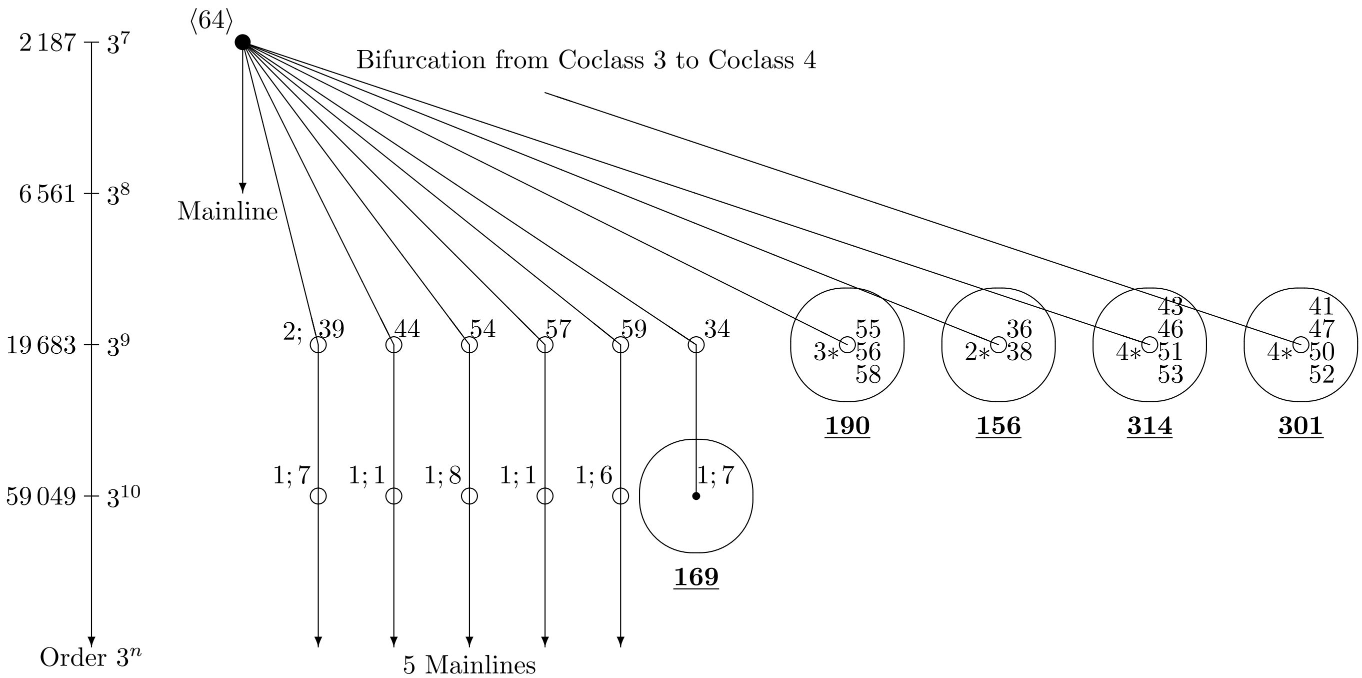
For identifying the groups $\mathrm{G}_3^2 K$ of type G.16, it was **not necessary** to compute Hilbert 3-class fields $\mathrm{F}_3^1 K$ of absolute degree 18.

§ 3. Statistics for $\mathrm{cc}(\mathrm{G}_3^2 K) = 4$

Table 6. 3-Capitulation Types F, G and H

Type	Artin Pattern		Freq.		#
	$\varkappa$	τ			
F.7∗	(3443)	$(32)^2, (1^3)^2$	190		3
F.11∗	(1143)	$(32)^2, (1^3)^2$	156		2
F.12∗	(1343)	$(32)^2, (1^3)^2$	314		4
F.13∗	(3143)	$(32)^2, (1^3)^2$	301		4
G.16∗	(1243)	$(32)^2, (1^3)^2$	70		1
G.19∗	(2143)	$(32)^2, (1^3)^2$	93		1
H.4∗	(4443)	$(32)^2, (1^3)^2$	169		1
F.7	(3443)	$43, 32, (1^3)^2$	26		4
F.11	(1143)	$43, 32, (1^3)^2$	25		2
F.12	(1343)	$43, 32, (1^3)^2$	70		8
F.13	(3143)	$43, 32, (1^3)^2$	66		8
G.16	(1243)	$43, 32, (1^3)^2$	13		1
G.19	(2143)	$43, 32, (1^3)^2$	21		1
H.4	(4443)	$43, 32, (1^3)^2$	27		1
F.7↑	(3443)	$54, 32, (1^3)^2$	2		4
F.11↑	(1143)	$54, 32, (1^3)^2$	1		2
F.12↑	(1343)	$54, 32, (1^3)^2$	9		8
F.13↑	(3143)	$54, 32, (1^3)^2$	1		8
H.4↑	(4443)	$54, 32, (1^3)^2$	3		1
		Total:	1 557	6.36% of 24 476	

FIGURE 10. Population of the sporadic coclass-4 graph $\mathcal{G}_0(3,4)$



Type	H.4∗	F.7∗	F.11∗	F.12∗	F.13∗
$\varkappa =$	(4443)	(3443)	(1143)	(1343)	(3143)

§ 4. Statistics for $\mathrm{cc}(\mathrm{G}_3^2 K) = 6$

TABLE 7. 3-Capitulation Types F and G

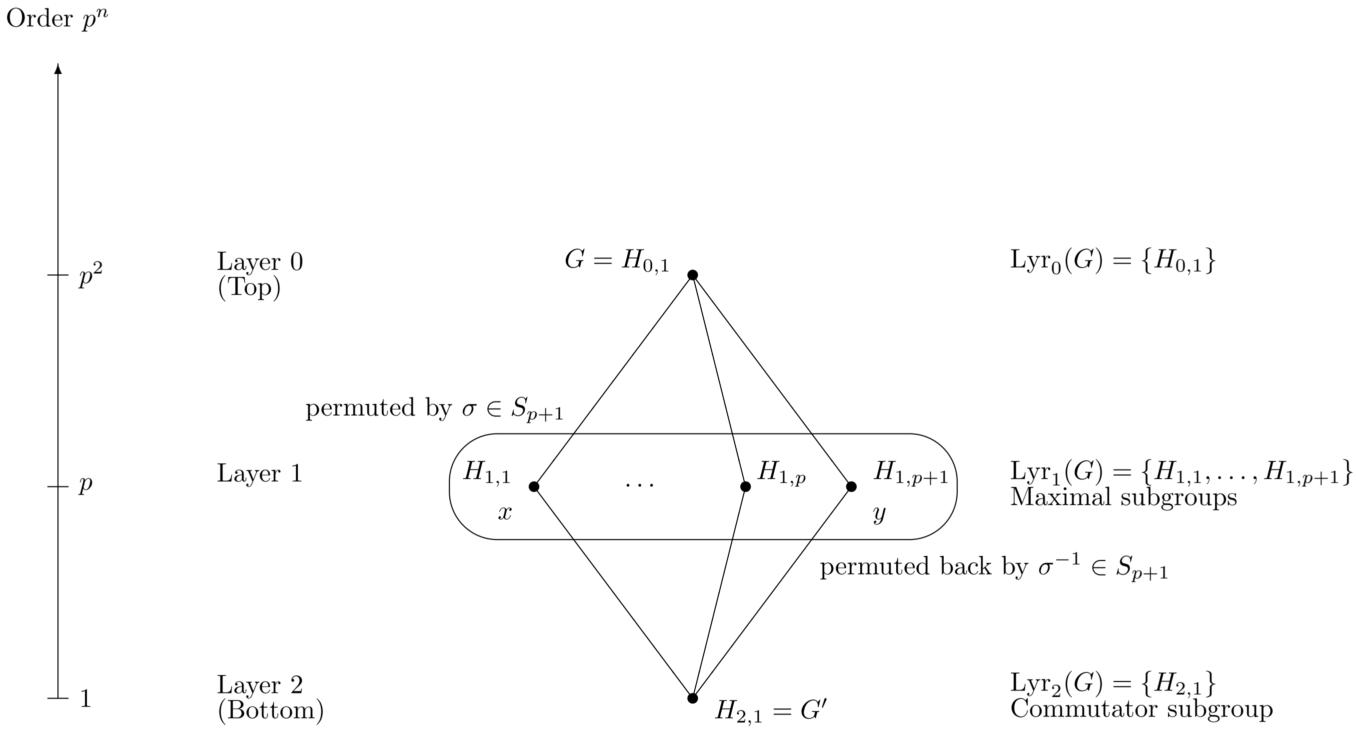
Type	Artin Pattern		Freq.		#
	$\varkappa$	τ			
F.7∗	(3443)	$(43)^2, (1^3)^2$	3		3
F.11∗	(1143)	$(43)^2, (1^3)^2$	3		2
F.12∗	(1343)	$(43)^2, (1^3)^2$	2		4
F.13∗	(3143)	$(43)^2, (1^3)^2$	6		4
G.19∗	(2143)	$(43)^2, (1^3)^2$	1		1
F.7	(3443)	$54, 43, (1^3)^2$	3		4
G.16	(1243)	$54, 43, (1^3)^2$	1		1
		Total:	19	0.08% of 24 476	

APPENDIX II

Brief p-Capitulation Types and their Orbits under Permutations

§ 1. Orbits of p-Capitulation Types

FIGURE 11. Subgroups of index, resp. order, p in $G/G' = \langle x, y \rangle \simeq (p,p)$



We have the correspondence
$H_{1,j}/G' \simeq \mathrm{Norm}_{L_j|K}\mathrm{Cl}_p L_j \quad \text{and} \quad G/G' \simeq \mathrm{Cl}_p K.$

For a **partial** capitulation, we put

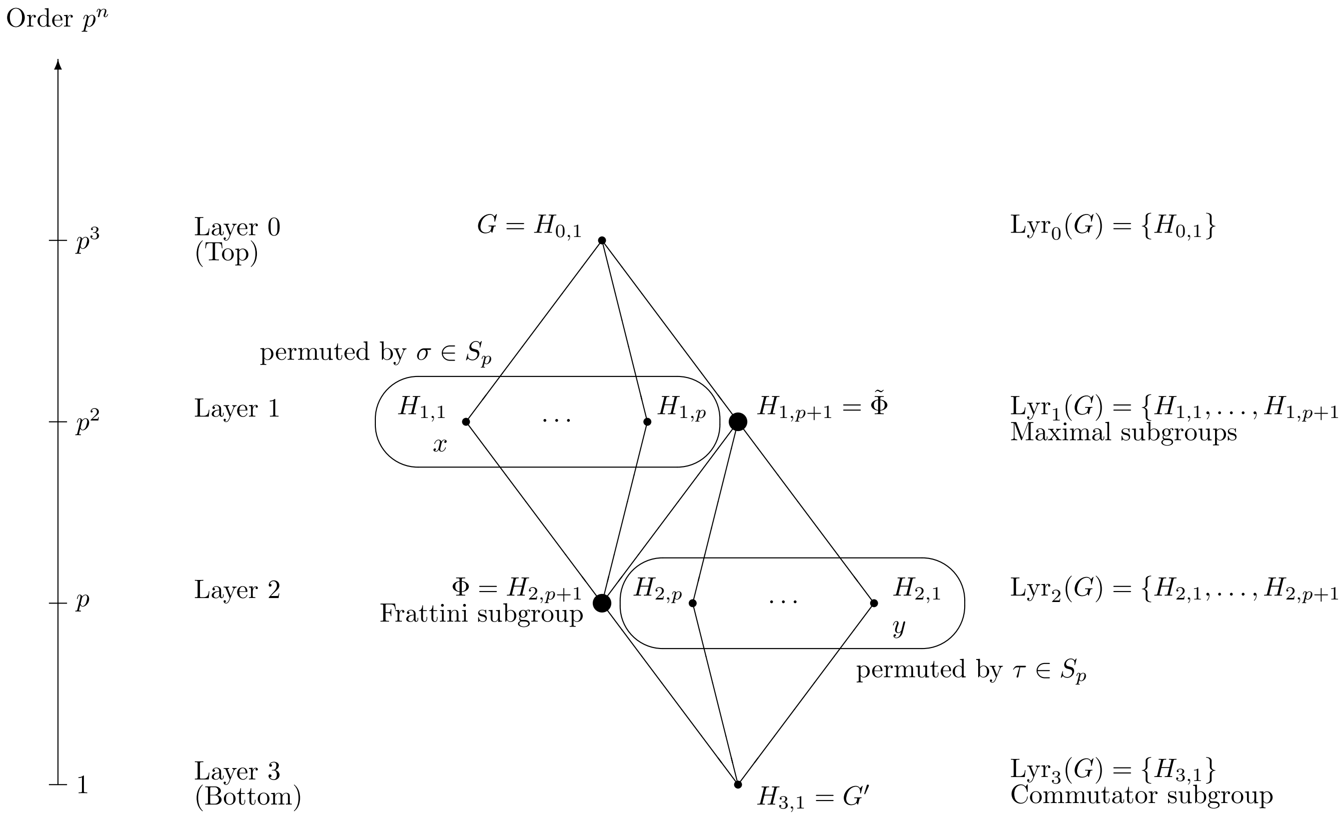
$$\ker J_{L_i|K} = \mathrm{Norm}_{L_j|K}\mathrm{Cl}_p L_j \iff: \varkappa_i(K) = j,$$

with some $1 \le j \le p+1$.

For a **total** capitulation, we put

$$\ker J_{L_i|K} = \mathrm{Cl}_p K = E_p \iff: \varkappa_i(K) = 0.$$

FIGURE 12. Subgroups of index, resp. order, p in $G/G' = \langle x,y\rangle \simeq (p^2,p)$



We have the correspondence
$H_{1,j}/G' \simeq \mathrm{Norm}_{L_j|K}\mathrm{Cl}_p L_j$ and $G/G' \simeq \mathrm{Cl}_p K$,
but the subgroups of $\tilde{\Phi} = H_{1,p+1}$ are $H_{2,1},\ldots,H_{2,p+1}$.

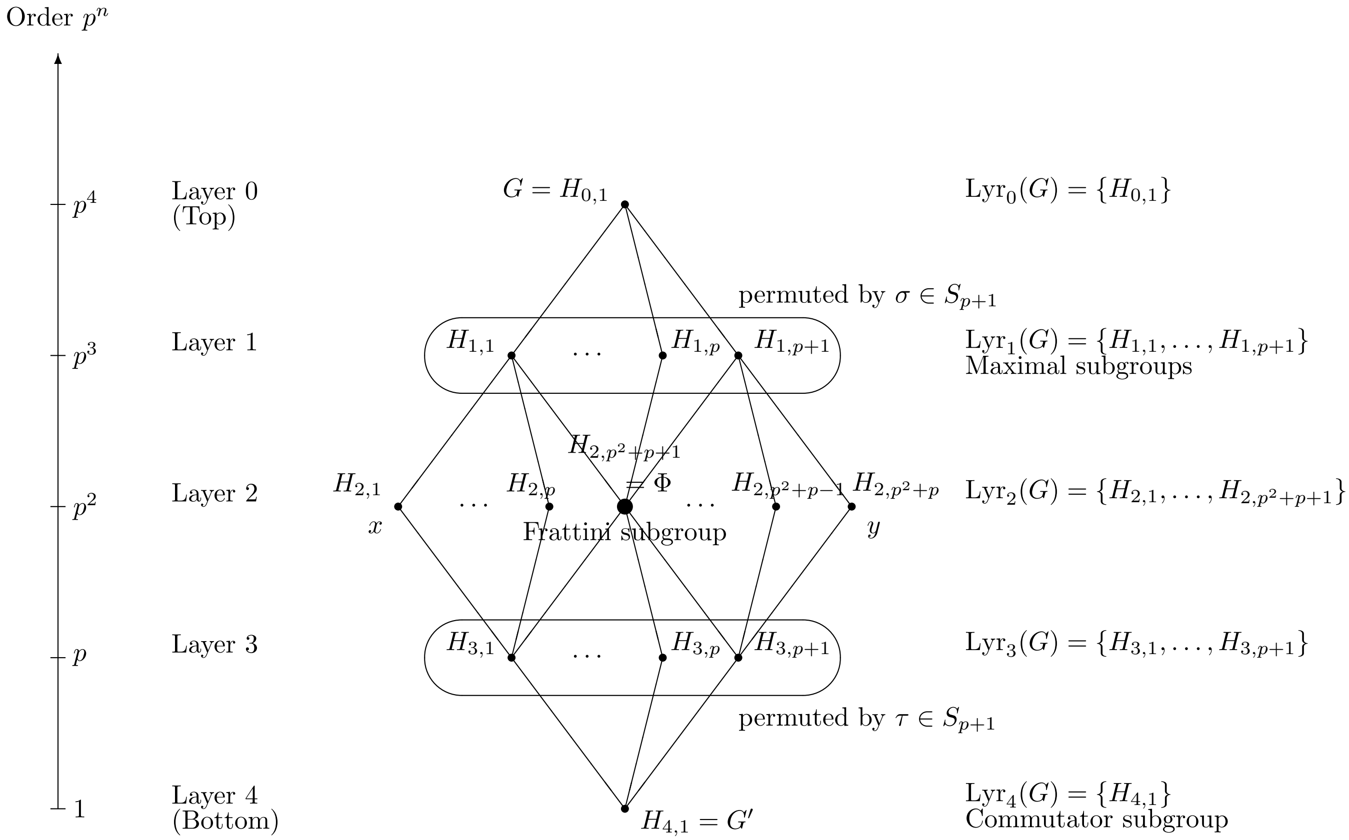
For a 1-**dimensional** capitulation, we put

$$\ker J_{L_i|K} \simeq H_{2,j}/G' \iff: \varkappa_i(K) = j,$$

with some $1 \le j \le p+1$.
For a 2-**dimensional** capitulation, we put

$$\ker J_{L_i|K} = E_p \simeq \tilde{\Phi}/G' \iff: \varkappa_i(K) = 0.$$

FIGURE 13. Subgroups of index, resp. order, p in $G/G' = \langle x, y\rangle \simeq (p^2, p^2)$



We have the correspondence
$H_{1,j}/G' \simeq \mathrm{Norm}_{L_j|K}\mathrm{Cl}_p L_j$ and $G/G' \simeq \mathrm{Cl}_p K$,
but the subgroups of $\Phi = H_{2,p^2+p+1}$ are $H_{3,1}, \ldots, H_{3,p+1}$.

For a 1-**dimensional** capitulation, we put

$$\ker J_{L_i|K} \simeq H_{3,j}/G' \iff: \varkappa_i(K) = j,$$

with some $1 \le j \le p+1$.
For a 2-**dimensional** capitulation, we put

$$\ker J_{L_i|K} = E_p \simeq \Phi/G' \iff: \varkappa_i(K) = 0.$$

§ 2. Orbits of 2-Capitulation Types

In this section, we put $p = 2$ and successively consider number fields K with 2-class group

$$\begin{aligned} \mathrm{Cl}_2 K &\simeq (2,2), \\ \mathrm{Cl}_2 K &\simeq (4,2), \\ \mathrm{Cl}_2 K &\simeq (4,4). \end{aligned}$$

With respect to orbit cardinalities, recall that $|S_3| = 6, \quad |S_2 \times S_2| = 4, \quad |S_3 \times S_3| = 36.$

2.1. Orbits under a single S_3-permutation $\varkappa^{S_3} = \{\sigma^{-1} \circ \varkappa \circ \sigma \mid \sigma \in S_3\}$

TABLE 8. 7 Orbits of 2-Capitulation Types $\varkappa \in [1,3]^3$ for $\mathrm{Cl}_2 K \simeq (2,2)$

No.	Type	Representative $\varkappa$	Occupation $o(\varkappa)$	Fixed Points	#
1	A.1	$(1,1,1)$	$(3,0,0)$	1	3
2	B.3	$(1,1,2)$	$(2,1,0)$	1	**6**
3	B.2	$(1,1,3)$	$(2,0,1)$	1, 3	**6**
4	Q.5	$(1,2,3)$	$(1,1,1)$	1, 2, 3	1
5	Q.6	$(1,3,2)$	$(1,1,1)$	1	3
6	S.4	$(2,1,1)$	$(2,1,0)$		**6**
7	C.7	$(2,3,1)$	$(1,1,1)$		2
				Total:	27

Arithmetical Realizations with $K = \mathbb{Q}(\sqrt{d})$, $d < 0$

$d = -120$: $\mathrm{Cl}_2 K \simeq (2,2)$, $G = Q(8)$, $\varkappa = (1,2,3)$, No. 4
quaternion

$d = -312$: $\mathrm{Cl}_2 K \simeq (2,2)$, $G = Q(16)$, $\varkappa = (1,3,2)$, No. 5
generalized quaternion

$d = -340$: $\mathrm{Cl}_2 K \simeq (2,2)$, $G = S(16)$, $\varkappa = (2,2,1)$, No. 6
semi-dihedral

No. 1, 2, 3, 7 are impossible, for group theoretic reasons.

2.2. Orbits under two independent S_2-permutations $\varkappa^{S_2 \times S_2} = \{\tau \circ \varkappa \circ \sigma \mid \sigma, \tau \in S_2, \sigma(3) = \tau(3) = 3\}$

TABLE 9. 10 Orbits of 2-Capitulation Types $\varkappa \in [1,3]^3$ for $\mathrm{Cl}_2K \simeq (2^u, 2)$, $u \geq 2$

No.	Type	Representative	Occupation	#
		$\varkappa$	$o(\varkappa)$	
1	A.1	$(1,1;1)$	$(3,0;0)$	2
2	B.2	$(1,1;2)$	$(2,1;0)$	2
3	B.3	$(1,1;3)$	$(2,0;1)$	2
4	B.4	$(1,2;1)$	$(2,1;0)$	**4**
5	C.5	$(1,2;3)$	$(1,1;1)$	2
6	B.6	$(1,3;1)$	$(2,0;1)$	**4**
7	C.7	$(1,3;2)$	$(1,1;1)$	**4**
8	B.8	$(1,3;3)$	$(1,0;2)$	**4**
9	B.9	$(3,3;1)$	$(1,0;2)$	2
10	A.10	$(3,3;3)$	$(0,0;3)$	1
			Total:	27

2.3. Orbits under two independent S_3-permutations $\varkappa^{S_3\times S_3}=\{\tau\circ\varkappa\circ\sigma\mid\sigma,\tau\in S_3\}$

Table 10. 3 Orbits of 2-Capitulation Types $\varkappa\in[1,3]^3$ for $\mathrm{Cl}_2K\simeq(2^u,2^v)$, $u,v\geq 2$

No.	Type	Representative	Occupation	#
		$\varkappa$	$o(\varkappa)$	
1	A	$(1,1,1)$	$(3,0,0)$	3
2	B	$(1,1,2)$	$(2,1,0)$	**18**
3	C	$(1,2,3)$	$(1,1,1)$	6
			Total:	27

§ 3. Orbits of 3-Capitulation Types

In this section, we put $p = 3$ and successively consider number fields K with 3-class group

$$\begin{aligned} \mathrm{Cl}_3 K &\simeq (3,3), \\ \mathrm{Cl}_3 K &\simeq (9,3), \\ \mathrm{Cl}_3 K &\simeq (9,9). \end{aligned}$$

With respect to orbit cardinalities, recall that $|S_4| = 24, \quad |S_3 \times S_3| = 36, \quad |S_4 \times S_4| = 576.$

3.1. Orbits under a single S_4-permutation $\varkappa^{S_4} = \{\sigma^{-1} \circ \varkappa \circ \sigma \mid \sigma \in S_4\}$

TABLE 11. 19 Orbits of 3-Capitulation Types $\varkappa \in [1,4]^4$ for $\mathrm{Cl}_3 K \simeq (3,3)$

No.	Type	Representative	Occupation	Fixed Points	#
		$\varkappa$	$o(\varkappa)$		
1	A.1	$(1,1,1,1)$	$(4,0,0,0)$	1	4
2	B.3	$(1,1,1,2)$	$(3,1,0,0)$	1	**24**
3	B.2	$(1,1,1,4)$	$(3,0,0,1)$	1, 4	12
4	E.6	$(1,1,2,2)$	$(2,2,0,0)$	1	12
5	D.10	$(1,1,2,3)$	$(2,1,1,0)$	1	**24**
6	E.9	$(1,1,2,4)$	$(2,1,0,1)$	1, 4	**24**
7	D.5	$(1,1,3,3)$	$(2,0,2,0)$	1, 3	12
8	E.8	$(1,1,3,4)$	$(2,0,1,1)$	1, 3, 4	12
9	F.11	$(1,1,4,3)$	$(2,0,1,1)$	1	12
10	C.15	$(1,2,3,4)$	$(1,1,1,1)$	1, 2, 3, 4	1
11	G.16	$(1,2,4,3)$	$(1,1,1,1)$	1, 2	6
12	F.12	$(1,3,2,2)$	$(1,2,1,0)$	1	**24**
13	C.17	$(1,3,4,2)$	$(1,1,1,1)$	1	8
14	H.4	$(2,1,1,1)$	$(3,1,0,0)$		12
15	F.7	$(2,1,1,2)$	$(2,2,0,0)$		12
16	F.13	$(2,1,1,3)$	$(2,1,1,0)$		**24**
17	G.19	$(2,1,4,3)$	$(1,1,1,1)$		3
18	E.14	$(2,3,1,1)$	$(2,1,1,0)$		**24**
19	C.18	$(2,3,4,1)$	$(1,1,1,1)$		6
				Total:	256

3.2. Orbits under two independent S_3-permutations $\varkappa^{S_3\times S_3} = \{\tau\circ\varkappa\circ\sigma \mid \sigma,\tau\in S_3, \sigma(4)=\tau(4)=4\}$

TABLE 12. 20 Orbits of 3-Capitulation Types $\varkappa\in[1,4]^4$ for $\mathrm{Cl}_3K\simeq(3^u,3)$, $u\ge 2$

No.	Type	Representative	Occupation	#
		$\varkappa$	$o(\varkappa)$	
1	A.1	$(1,1,1;1)$	$(4,0,0;0)$	3
2	B.2	$(1,1,1;2)$	$(3,1,0;0)$	6
3	B.7	$(1,1,1;4)$	$(3,0,0;1)$	3
4	B.3	$(1,1,2;1)$	$(3,1,0;0)$	18
5	C.4	$(1,1,2;2)$	$(2,2,0;0)$	18
6	D.5	$(1,1,2;3)$	$(2,1,1;0)$	18
7	D.9	$(1,1,2;4)$	$(2,1,0;1)$	18
8	B.8	$(1,1,4;1)$	$(3,0,0;1)$	9
9	D.10	$(1,1,4;2)$	$(2,1,0;1)$	18
10	C.14	$(1,1,4;4)$	$(2,0,0;2)$	9
11	D.6	$(1,2,3;1)$	$(2,1,1;0)$	18
12	E.12	$(1,2,3;4)$	$(1,1,1;1)$	6
13	D.11	$(1,2,4;1)$	$(2,1,0;1)$	**36**
14	E.13	$(1,2,4;3)$	$(1,1,1;1)$	18
15	D.16	$(1,2,4;4)$	$(1,1,0;2)$	18
16	C.15	$(1,4,4;1)$	$(2,0,0;2)$	9
17	D.17	$(1,4,4;2)$	$(1,1,0;2)$	18
18	B.18	$(1,4,4;4)$	$(1,0,0;3)$	9
19	B.19	$(4,4,4;1)$	$(1,0,0;3)$	3
20	A.20	$(4,4,4;4)$	$(0,0,0;4)$	1
			Total:	256

3.3. Orbits under two independent S_4-permutations $\varkappa^{S_4\times S_4}=\lbrace\tau\circ\varkappa\circ\sigma\mid\sigma,\tau\in S_4\rbrace$

TABLE 13. 5 Orbits of 3-Capitulation Types $\varkappa\in[1,4]^4$ for $\mathrm{Cl}_3K\simeq(3^u,3^v)$, $u,v\ge 2$

No.	Type	Representative	Occupation	#
		$\varkappa$	$o(\varkappa)$	
1	A	$(1,1,1,1)$	$(4,0,0,0)$	4
2	B	$(1,1,1,2)$	$(3,1,0,0)$	48
3	C	$(1,1,2,2)$	$(2,2,0,0)$	36
4	D	$(1,1,2,3)$	$(2,1,1,0)$	**144**
5	E	$(1,2,3,4)$	$(1,1,1,1)$	24
			Total:	256

Arithmetical Realizations with $K=\mathbb{Q}(\sqrt{d})$, $d<0$

$d=-134\,059$: $\mathrm{Cl}_3K\simeq(9,9)$, $\varkappa=(3,2,4,1)$, No. 5

$d=-208\,084$: $\mathrm{Cl}_3K\simeq(9,9)$, $\varkappa=(2,2,2,2)$, No. 1

$d=-5\,184\,227$: $\mathrm{Cl}_3K\simeq(81,9)$, $\varkappa=(3,3,1,3)$, No. 2

§ 4. Orbits of 5-Capitulation Types

In this section, we put $p = 5$ and successively consider number fields K with 5-class group

$$\mathrm{Cl}_5 K \simeq (5,5),$$
$$\mathrm{Cl}_5 K \simeq (25,5),$$
$$\mathrm{Cl}_5 K \simeq (25,25).$$

With respect to orbit cardinalities, recall that
$|S_6| = 720, \quad |S_5 \times S_5| = 14\,400,$
$|S_6 \times S_6| = 518\,400.$

4.1. Orbits under a single S_6-permutation $\varkappa^{S_6} = \{\sigma^{-1} \circ \varkappa \circ \sigma \mid \sigma \in S_6\}$

TABLE 14. 130 Orbits of 5-Capitulation Types $\varkappa \in [1,6]^6$ for $\mathrm{Cl}_5 K \simeq (5,5)$

No.	Type	Representative	Occupation	Fixed Points	#
		$\varkappa$	$o(\varkappa)$		
1		$(1,1,1,1,1,1)$	$(6,0,0,0,0,0)$	1	6
2		$(1,1,1,1,1,2)$	$(5,1,0,0,0,0)$	1	120
3		$(1,1,1,1,1,6)$	$(5,0,0,0,0,1)$	1, 6	30
4		$(1,1,1,1,2,2)$	$(4,2,0,0,0,0)$	1	180
5		$(1,1,1,1,2,3)$	$(4,1,1,0,0,0)$	1	360
6		$(1,1,1,1,2,5)$	$(4,1,0,0,1,0)$	1	360
7		$(1,1,1,1,2,6)$	$(4,1,0,0,0,1)$	1, 6	360
8		$(1,1,1,1,5,5)$	$(4,0,0,0,2,0)$	1, 5	120
9		$(1,1,1,1,5,6)$	$(4,0,0,0,1,1)$	1, 5, 6	60
10		$(1,1,1,1,6,5)$	$(4,0,0,0,1,1)$	1, 6	60
		$\dots$	$\dots$		
128		$(2,3,4,1,1,5)$	$(2,1,1,1,1,0)$		**720**
129		$(2,3,4,5,1,1)$	$(2,1,1,1,1,0)$		**720**
130		$(2,3,4,5,6,1)$	$(1,1,1,1,1,1)$		120
				Total:	46 656

4.2. Orbits under two independent S_5-permutations $\varkappa^{S_5\times S_5} = \{\tau\circ\varkappa\circ\sigma \mid \sigma,\tau\in S_5, \sigma(6)=\tau(6)=6\}$

TABLE 15. 63 Orbits of 5-Capitulation Types $\varkappa\in[1,6]^6$ for $\mathrm{Cl}_5K\simeq(5^u,5)$, $u\ge 2$

No.	Type	Representative	Occupation	#
		$\varkappa$	$o(\varkappa)$	
1		$(1,1,1,1,1;1)$	$(6,0,0,0,0;0)$	5
2		$(1,1,1,1,1;2)$	$(5,1,0,0,0;0)$	20
3		$(1,1,1,1,1;6)$	$(5,0,0,0,0;1)$	5
4		$(1,1,1,1,2;1)$	$(5,1,0,0,0;0)$	100
5		$(1,1,1,1,2;2)$	$(4,2,0,0,0;0)$	100
6		$(1,1,1,1,2;3)$	$(4,1,1,0,0;0)$	300
7		$(1,1,1,1,2;6)$	$(4,1,0,0,0;1)$	100
8		$(1,1,1,1,6;1)$	$(5,0,0,0,0;1)$	25
9		$(1,1,1,1,6;2)$	$(4,1,0,0,0;1)$	100
10		$(1,1,1,1,6;6)$	$(4,0,0,0,0;2)$	25
		$\dots$	$\dots$	
34		$(1,1,2,3,4;2)$	$(2,2,1,1,0;0)$	**3 600**
		$\dots$	$\dots$	
61		$(1,6,6,6,6;6)$	$(1,0,0,0,0;5)$	25
62		$(6,6,6,6,6;1)$	$(1,0,0,0,0;5)$	5
63		$(6,6,6,6,6;6)$	$(0,0,0,0,0;6)$	1
			Total:	46 656

4.3. Orbits under two independent S_6-permutations $\varkappa^{S_6 \times S_6} = \{\tau \circ \varkappa \circ \sigma \mid \sigma, \tau \in S_6\}$

TABLE 16. 11 Orbits of 5-Capitulation Types $\varkappa \in [1,6]^6$ for $\mathrm{Cl}_5 K \simeq (5^u, 5^v)$, $u, v \geq 2$

No.	Type	Representative	Occupation	#
		$\varkappa$	$o(\varkappa)$	
1		$(1,1,1,1,1,1)$	$(6,0,0,0,0,0)$	6
2		$(1,1,1,1,1,2)$	$(5,1,0,0,0,0)$	180
3		$(1,1,1,1,2,2)$	$(4,2,0,0,0,0)$	450
4		$(1,1,1,1,2,3)$	$(4,1,1,0,0,0)$	1 800
5		$(1,1,1,2,2,2)$	$(3,3,0,0,0,0)$	300
6		$(1,1,1,2,2,3)$	$(3,2,1,0,0,0)$	7 200
7		$(1,1,1,2,3,4)$	$(3,1,1,1,0,0)$	7 200
8		$(1,1,2,2,3,3)$	$(2,2,2,0,0,0)$	1 800
9		$(1,1,2,2,3,4)$	$(2,2,1,1,0,0)$	**16 200**
10		$(1,1,2,3,4,5)$	$(2,1,1,1,1,0)$	10 800
11		$(1,2,3,4,5,6)$	$(1,1,1,1,1,1)$	720
			Total:	46 656