New perspectives of the power-commutator structure: Coclass trees of CF-groups and related BCF-groups

Daniel C. Mayer

Abstract Let $e \geq 2$ be an integer. Among the finite 3-groups G with bicyclic commutator quotient $G/G' \simeq (\mathbb{Z}/3^e\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$, having one non-elementary component with logarithmic exponent e, there exists a unique pair of coclass trees with distinguished rank distribution $\varrho \sim (2,2,3;3)$. One tree $\mathcal{T}(C_1^{(e)})$ consists of CF-groups with coclass e, and the other tree $\mathcal{T}(B_1^{(e+1)})$ consists of BCF-groups with coclass $e+1$. We prove that, due to a chain of periodic bifurcations, the vertices of all pairs $(\mathcal{T}(C_1^{(e)}), \mathcal{T}(B_1^{(e+1)}))$ with $e \geq 3$ can be constructed as p-descendants $C_1^{(e+1)} = C_1^{(e)} - \#1;1$ and $B_1^{(e+1)} = C_1^{(e)} - \#1;2$ of the single root $C_1^{(3)} = \mathrm{SmallGroup}(3^6, 7)$ recursively with the aid of the p-group generation algorithm by Newman and O'Brien.

1 Preface and Introduction

In the past decade, we have realized that the identification of the stages $K \leq \mathrm{F}_p^1(K) \leq \mathrm{F}_p^2(K) \leq \mathrm{F}_p^3(K) \leq \ldots$ of the Hilbert p-class field tower $\mathrm{F}_p^\infty(K)$, that is, the maximal unramified pro-p-extension, of an algebraic number field K with bicyclic p-class group $\mathrm{Cl}_p(K) = \mathrm{Syl}_p\mathrm{Cl}(K) \simeq (\mathbb{Z}/p^e\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$, possessing one non-elementary component with $e \geq 2$, poses unexpected group-theoretical challenges which were unknown in the elementary bicyclic situation with $e = 1$. Whereas the simplest case $e = 2$ is still rather well-behaved, there arise increasing difficulties (§ 12) in the construction of the required inventory of finite p-groups as candidates for the groups $\mathrm{Gal}(\mathrm{F}_p^n(K)/K)$ of the tower stages for growing exponents $e \geq 3$. Firstly, the recursive construction of descendant trees of finite p-groups

Daniel C. Mayer
Naglergasse 53
8010 Graz
Austria, e-mail: algebraic.number.theory@algebra.at

G with commutator quotient $G/G' \simeq (\mathbb{Z}/p^e\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$ starting at an abelian root by repeated applications of the p-group generation algorithm by Newman and O'Brien [1, 2] becomes impossible for nilpotency class c smaller than the exponent e, because the trees are defined by relations $P = D/\gamma_c(D)$ between parent P and descendant D, in terms of the *lower central series* $(\gamma_i(G))_{1\le i\le c+1}$ with nilpotency class $c = \mathrm{cl}(G)$ expressing the *commutator structure*, but the algorithm is based on the *exponent-p-central series* $(P_i(G))_{0\le i\le c_p}$ with p-class $c_p = \mathrm{cl}_p(G)$ dominated by the *power structure* [3, pp. 176–177]. Secondly, two important kinds of groups, the *CF-groups* [4] with cyclic factors of the lower central series and the *BCF-groups* [5] with bicyclic or cyclic factors, which are completely independent for $e \in \{1, 2\}$, become connected by relations $A = S/P_{c_p-1}(S)$ between ancestor A and successor S, for $e \ge 3$.

Therefore, in order to arrive at new horizons in algebraic number theory and class field theory it is necessary to establish new perspectives of the power-commutator structure of finite p-groups.

In a series of preceding papers [6, 7, 8, 9], we have developed a new theory of finite 3-groups G with bicyclic commutator quotient $G/G' \simeq C_{3^e} \times C_3$, having one non-elementary component with logarithmic exponent $e \ge 2$. Theoretical foundations were based on two invariants of G with respect to its four maximal subgroups $H_1, H_2, H_3; H_4$ (with distinguished H_4), the abelian quotient invariants (AQI) $\alpha(G) = (H_i/H_i')_{i=1}^4$ and the punctured transfer kernel type (pTKT) $\varkappa(G) = (\ker(T_i))_{i=1}^4$, combined in the *Artin pattern* $\mathrm{AP}(G) = (\alpha(G), \varkappa(G))$ [9, pp. 149–151].

The primary motivation for these papers was the application to possible automorphism groups $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of 3-class field towers over *imaginary* quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $-10^6 < d < 0$, which must be *Schur σ-groups* (with balanced presentation and generator inverting (GI) automorphism). The first systematic investigation of this kind of number fields with 3-class group $\mathrm{Cl}_3(K) \simeq (9, 3)$ was conducted by ourselves in 2011, presented in [10], and summarized in [9]. In the justification of newly discovered *periodicities* among Schur σ-groups, two strange phenomena, mentioned at the beginning already, attracted our vigilance and attention:

- cumbersome difficulties in the construction of groups with small nilpotency class $\mathrm{cl}(G) \le e$,
- unexpected connections and relationships between CF-groups [4] and BCF-groups [5].

In the present work, we abandon all motivations by algebraic number theory and class field theory, we remove the focus on Schur groups and even on σ-groups (except in § 10), and we solve the above mentioned two problems completely for two infinite families of *coclass trees* [3, 11], one, $\mathcal{T}^e(C_1^{(e)})$, consisting of CF-groups and mainline of type $\mathrm{a}.1$, the other, $\mathcal{T}^{e+1}(B_1^{(e+1)})$, consisting of BCF-groups and mainline of type $\mathrm{d}.10$, and unbounded exponent $e \ge 3$ in both situations.

The first difficulty is explained by shedding new light on the *commutator structure* and *power structure* and their impact on the descending central series, the lower exponent-p-central series, and the p-group generation algorithm [12, 13, 1, 2] (also called *extension algorithm* in [4]).

The second phenomenon is due to closely related *power-commutator-presentations* for certain CF-groups and BCF-groups, the *mainline principle* for the generator of the last non-trivial lower central $\gamma_c(G) = \langle s_c \rangle$, and peculiarities of the last non-trivial lower p-central $P_{c_p-1}(G) = \langle w \rangle$.

The marvellous and astonishing statement of Theorems 1 and 2 is the constructibility of all vertices $C_{i,j}^{(e)}$, $i, j \geq 1$, on infinitely many CF-coclass trees $\mathcal{T}^e(C_1^{(e)})$, $e \geq 3$, of type $\mathrm{a}.1$, $\varkappa = (0, 0, 0; 0)$, with rank distribution $\varrho \sim (2, 2, 3; 3)$, as descendants of a single root $C_1^{(3)} = \langle 729, 7 \rangle$ [14], which is the analogue of **Ascione's CF-group A** [4] for the commutator quotient $(27, 3)$. The highlight of this work, completely unexpected up to now, asserts the constructibility of all vertices $B_{i,j}^{(e+1)}$, $i, j \geq 1$, on infinitely many BCF-coclass trees $\mathcal{T}^{e+1}(B_1^{(e+1)})$, $e \geq 3$, of type $\mathrm{d}.10$, $\varkappa = (1, 1, 0; 2)$, also with rank distribution $\varrho \sim (2, 2, 3; 3)$, as descendants of the same CF-root $C_1^{(3)} = \langle 729, 7 \rangle$, according to Theorem 3 and Algorithm 1.

2 Foundations and Layout of this Work

We consider finite 3-groups G whose commutator quotient is *bicyclic* with a *single non-elementary component*, that is, $G/G' \simeq C_{3^e} \times C_3$ with logarithmic exponent $e \geq 2$. By the Burnside basis theorem, $G = \langle x, y \rangle$ is two-generated, and we stick to the convention for the generators that $w = x^{3^{e-1}}$, $w^3 \in G'$ and $y^3 \in G'$.

For such groups, we have introduced several invariants [9] in terms of their maximal normal subgroups $H_1, H_2, H_3; H_4$ of index $(G : H_i) = 3$, where the puncture at the fourth component is motivated by the distinction of the bicyclic quotient $H_4/G' \simeq C_{3^{e-1}} \times C_3$, as opposed to the cyclic quotients $H_i/G' \simeq C_{3^e}$ for $1 \leq i \leq 3$. We have the logarithmic *abelian quotient invariants* (AQI),

$$\alpha(G) = \big(H_1/H_1', H_2/H_2', H_3/H_3'; H_4/H_4'\big), \tag{1}$$

their *rank distribution*

$$\varrho(G) = \big(\mathrm{rank}_3(H_1/H_1'), \mathrm{rank}_3(H_2/H_2'), \mathrm{rank}_3(H_3/H_3'); \mathrm{rank}_3(H_4/H_4')\big), \tag{2}$$

and, on the other hand, the *punctured transfer kernel type* (pTKT),

$$\varkappa(G) = \big(\ker(T_1), \ker(T_2), \ker(T_3); \ker(T_4)\big), \tag{3}$$

where $T_i : G/G' \to H_i/H_i'$ denotes the Artin transfer homomorphism from G to H_i. AQI and pTKT are combined in the *Artin pattern* $\mathrm{AP}(G) = (\alpha(G), \varkappa(G))$ of G. Since there are only five possibilities for the kernels, the pTKT is abbreviated in the form $\varkappa(G) = \big(\varkappa_1, \varkappa_2, \varkappa_3; \varkappa_4\big)$, where

$$\varkappa_i = \begin{cases} 0 \text{ if } \ker(T_i) = \langle w, y\rangle/G' \text{ (complete 3-elementary subgroup)}, \\ j \text{ if } \ker(T_i) = \langle w^{j-1}\cdot y\rangle/G',\ 1 \le j \le 3, \\ 4 \text{ if } \ker(T_i) = \langle w\rangle/G' \text{ (distinguished third power generator)}. \end{cases} \quad (4)$$

Our special attention is devoted to *CF-groups* G for which the factors $\gamma_i(G)/\gamma_{i+1}(G)$, $i \ge 2$, of the descending central series $(\gamma_i(G))_{i\ge 1}$ are cyclic of order 3 (*cyclic factor groups*). Let $s_2 = [y, x]$, $s_3 = [s_2, x]$, and $t_3 = [s_2, y]$ denote essential commutators. Since $\gamma_2(G) = \langle s_2, \gamma_3(G)\rangle$, the second factor is always cyclic, but since $\gamma_3(G) = \langle s_3, t_3, \gamma_4(G)\rangle$, the third factor is usually bicyclic (*BCF — bicyclic or cyclic factor groups*), and there must exist some relation between s_3 and t_3 in a CF-group, for instance, either one of the two commutators is trivial or $s_3 = t_3$.

Even more specifically, our focus will lie on *coclass trees* [15, p. 89] whose vertices G share the common rank distribution $\varrho(G) \sim (223; 3)$, that is, trees of CF-groups with mainline of type $\mathrm{a}.1$, $\varkappa(G) = (000; 0)$, and trees of BCF-groups with mainline of type $\mathrm{d}.10$, $\varkappa(G) \sim (110; 2)$.

After a summary of foundations in § 3, we begin with simple laws for all mainlines of CF-coclass trees in § 4. Then we extend the investigations to chains of periodic bifurcations in § 5, where the complete system of all CF- and BCF-coclass trees with $e \ge 3$ is shown to arise from a single root.

3 Basic Definitions and Conventions

The lower exponent-p-central series of a finite 3-group G will always be denoted by $(P_i(G))_{i\ge 0}$.

Definition 1 Let D be non-trivial finite 3-group with nilpotency class $c = \mathrm{cl}(D) \ge 1$ and lower exponent p-class $c_p = \mathrm{cl}_p(D) \ge 1$, i.e., $\gamma_c(D) > \gamma_{c+1}(D) = 1$ and $P_{c_p-1}(D) > P_{c_p}(D) = 1$. Then the quotient $A = \pi(D) = D/\gamma_c(D)$ is called the *parent* of D and the quotient $A_p = \pi_p(D) = D/P_{c_p-1}(D)$ is called the *p-parent* of D. Conversely, D is called an *immediate descendant* of A and an *immediate p-descendant* of A_p. By the *root path*, respectively *p-root path*, of D we understand the sequence $(\pi^i(D))_{i=0}^{c-1}$, respectively $(\pi_p^i(D))_{i=0}^{c_p-1}$, of its iterated parents, respectively p-parents.

Definition 2 The *propagation* from a p-parent A to a p-descendant D is called **endo-genetic** if the commutator quotient remains unchanged, that is, $D/D' \simeq A/A'$. Otherwise the propagation is called **exo-genetic**.

The propagation from non-trivial parent A to non-abelian descendant D is always endo-genetic, because $A = D/\gamma_c(D)$ with $c \ge 2$, and thus $D/\gamma_2(D) \simeq (D/\gamma_c(D))/(\gamma_2(D)/\gamma_c(D)) \simeq A/\gamma_2(A)$.

Definition 3 The *descendant tree* $\mathcal{T}(R)$, respectively p-*descendant tree* $\mathcal{T}_p(R)$, with a finite 3-group R as its root consists of the following vertices and directed edges: the vertices are all isomorphism classes of finite 3-groups D whose root path, respectively p-root path, contains R, and the directed edges are all pairs (D, A), also denoted by $D \to A$, of immediate descendants D and parents $A = \pi(D)$, respectively p-parents $A = \pi_p(D)$, among the vertices of the tree. A descendant tree whose vertices are subject to certain restrictive conditions is called a *pruned tree*.

Definition 4 A pruned tree which contains a unique infinite *mainline* and all of whose vertices share a common coclass r is called a *coclass tree*. If the root is R, the tree is denoted by $\mathcal{T}^r(R)$.

The step size of all edges in a coclass tree is necessarily $s = 1$. *Depth-pruned* branches of a coclass tree become periodic, beginning with a minimal *periodic root* on the mainline [15, Thm. 3.1].

Definition 5 By a *tree of type* X we understand a coclass tree whose mainline consists of vertices with (punctured) transfer kernel type X [9, Tbl. 1–2, pp. 3–4]. (For instance $\mathrm{X} = \mathrm{a}.1$ or $\mathrm{d}.10$.)

We introduce an ostensive terminology in order to illuminate three distinct situations with crucial differences in the construction by means of the p-group generation algorithm [12, 13, 1, 2].

Definition 6 A vertex V on a coclass tree $\mathcal{T}^r$, with $V/V' \simeq C_{3^e} \times C_3$ and $r \in \{e, e+1\}$, lies

- **behind the shockwave**, if $\mathrm{cl}(V) < r$,
- **on the shockwave**, if $\mathrm{cl}(V) = r$,
- **ahead of the shockwave**, if $\mathrm{cl}(V) > r$.

The behavior ahead of the shockwave will turn out to be *regular* with endo-genetic propagation, dominated by the *commutator structure*. In contrast, we shall see that the behavior behind the shockwave is *irregular* with exo-genetic propagation, due to a dominance of the *power structure*. A *singular* behavior can be observed on the shockwave, where the propagation is mixed, partially endo-genetic and partially exo-genetic, and *periodic bifurcations* arise, because both, the commutator structure and the power structure, exert a combined impact.

In order to identify the isomorphism class of a finite 3-group G, several ways are possible.
Either the group is characterized by its *absolute identifier* $\mathrm{SmallGroup}(o, i)$, or briefly $\langle o, i\rangle$, in the SmallGroups database [14], where $o = \#G$ denotes the order of G, bounded by $o \le 3^8$, and i is a positive integer. The short form in angle brackets is returned by the Magma statement `IdentifyGroup()` [16, 17, 18], provided that

$o \leq 3^6$. When the order $o = 3^e$ or the logarithmic order e is given along a scale on the left hand side of a figure illustrating a descendant tree of finite 3-groups, then we omit the order o in the absolute identifier $\langle o, i \rangle$ and simply write $\langle i \rangle$.

Or G is constructed by means of `Descendants(`P`:StepSizes:=[`s`])` as an immediate step size-s p-descendant of a p-parent group P and characterized by a *relative identifier* $G = P - \#s; j$ with $1 \leq s \leq n(P)$ and $1 \leq j \leq N_s(P)$, where $n(P)$ denotes the nuclear rank of the p-parent P and $N_s(P)$ is the number of immediate step size-s p-descendants of P.

Finally, there is always the possibility to give a *power commutator (pc-) presentation* for G.

4 Laws for Coclass Trees of CF-Groups

We separate our main statements into three parts: uniqueness, invariants, and construction.

Proposition 1 ***(Uniqueness)*** *For each logarithmic exponent $e \geq 2$, there exists a* ***unique*** *coclass tree $\mathcal{T}^e(C_1^{(e)}) \ni V$ with fixed coclass $\mathrm{cc}(V) = e$, fixed commutator quotient $V/V' \simeq C_{3^e} \times C_3$, and fixed rank distribution $\varrho(V) \sim (2,2,3;3)$. Its mainline $(C_i^{(e)})_{i\geq 1}$ is of type $\mathrm{a}.1$, $\varkappa(C_i^{(e)}) = (0,0,0;0)$. The tree consists* ***entirely*** *of* ***metabelian*** *CF-groups. The branches are of depth 3. (See Figures 2–4 for $2 \leq e \leq 4$.)*

Proof For each commutator quotient $C_{3^e} \times C_3$ with log exponent $e \geq 2$, there exists a *finite* number N of coclass trees $\mathcal{T}^r(R_j^{(r)})$ with roots $R_j^{(r)}$, $1 \leq j \leq N$, and two minimal possible values $e \leq r \leq e+1$ for the coclass. Descendant vertices V of each root share invariants with the root, e.g. the rank distribution $\varrho(V)$. The roots with $r = e+1$ are non-CF groups (called *BCF-groups* in [5], i.e. groups with *bicyclic or cyclic factors* of the lower central series), and the others with $r = e$ are CF-groups. There are only two trees with rank distribution $\varrho(V) \sim (2,2,3;3)$, a BCF-tree of type $\mathrm{d}.10$ and a CF-tree of type $\mathrm{a}.1$. The latter is the *unique* tree with root $R_j^{(e)} = C_1^{(e)}$, recursively determined by $C_1^{(3)} = \langle 729, 7 \rangle$ and $C_1^{(2)} = \langle 243, 17 \rangle$, according to Theorems 1 and 2. Its branches are periodic of length 2 without preperiod, and all of its vertices are metabelian CF-groups, since the vertices of the first two branches are metabelian CF-groups. □

4.1 Vertices on the Mainline (with Depth 0)

Proposition 2 ***(Invariants)*** *For $e \geq 3$, invariants of vertices on the mainline $(C_i^{(e)})_{i\geq 1}$ of the unique CF-coclass tree $\mathcal{T}^e(C_1^{(e)})$ are given as follows:*

$$\begin{aligned}
&\textit{logarithmic order } \mathrm{lo}(C_i^{(e)}) = e+i+2, \textit{ for } i \ge 1,\\
&\textit{nilpotency class } \mathrm{cl}(C_i^{(e)}) = i+2, \textit{ for } i \ge 1,\\
&p\textit{-class } \mathrm{cl}_p(C_i^{(e)}) = \begin{cases} i+2 & \textit{if } i > e-2,\\ e & \textit{if } i \le e-2,\end{cases}\\
&p\textit{-coclass } \mathrm{cc}_p(C_i^{(e)}) = \begin{cases} e & \textit{if } i > e-2,\\ i+2 & \textit{if } i \le e-2.\end{cases}
\end{aligned} \tag{5}$$

Proof Proposition 2 remains true when the mainline vertex $C_i^{(e)}$ is replaced by any proper descendant vertex $C_{i,j}^{(e)}$ with $i \ge 2$. All coclass trees under investigation start at a root of class $\mathrm{cl}(C_1^{(e)}) = 3 = 1+2$, for each $e \ge 2$. Thus, proper descendants possess nilpotency class $\mathrm{cl}(C_{i,j}^{(e)}) = i+2 \ge 4$. By definition, all vertices V of the coclass tree $\mathcal{T}^e(C_1^{(e)})$ share the common coclass $\mathrm{cc}(V) = e$. Consequently, the logarithmic order is the sum $\mathrm{lo}(C_{i,j}^{(e)}) = \mathrm{cl}(C_{i,j}^{(e)}) + \mathrm{cc}(C_{i,j}^{(e)}) = i+2+e$. Finally, the *power structure* of all finite 3-groups G with commutator quotient $G/G' \simeq C_{3^e} \times C_3$ is responsible for the constant p-class $\mathrm{cl}_p(C_{i,j}^{(e)}) = e$, independently of the class $\mathrm{cl}(C_{i,j}^{(e)}) = i+2 \le e$, in the finite region on and behind the shock wave. □

Theorem 1 ***(Construction)*** *Vertices on the mainline* $(C_i^{(e)})_{i\ge 1}$ *of the coclass tree* $\mathcal{T}^e(C_1^{(e)})$ *can be constructed recursively, according to three laws in dependence on the nilpotency class,*

- *by horizontal* ***irregular exo****-genetic propagation (behind the shockwave)*

$$\pi_p(C_i^{(e)}) = C_i^{(e-1)}, \quad C_i^{(e)} = C_i^{(e-1)} - \#1;1, \tag{6}$$

  for $e \ge 4,\ i \le e-3$*, i.e.* $\mathrm{cl}(C_i^{(e)}) < e$*,*
- *by diagonal* ***singular exo****-genetic propagation (bifurcation on the shockwave)*

$$\pi_p(C_i^{(e)}) = C_{i-1}^{(e-1)}, \quad C_i^{(e)} = C_{i-1}^{(e-1)} - \#2;1, \tag{7}$$

  for $e \ge 4,\ i = e-2$*, i.e.* $\mathrm{cl}(C_i^{(e)}) = e$*,*
- *by vertical* ***regular endo****-genetic propagation (ahead of the shockwave)*

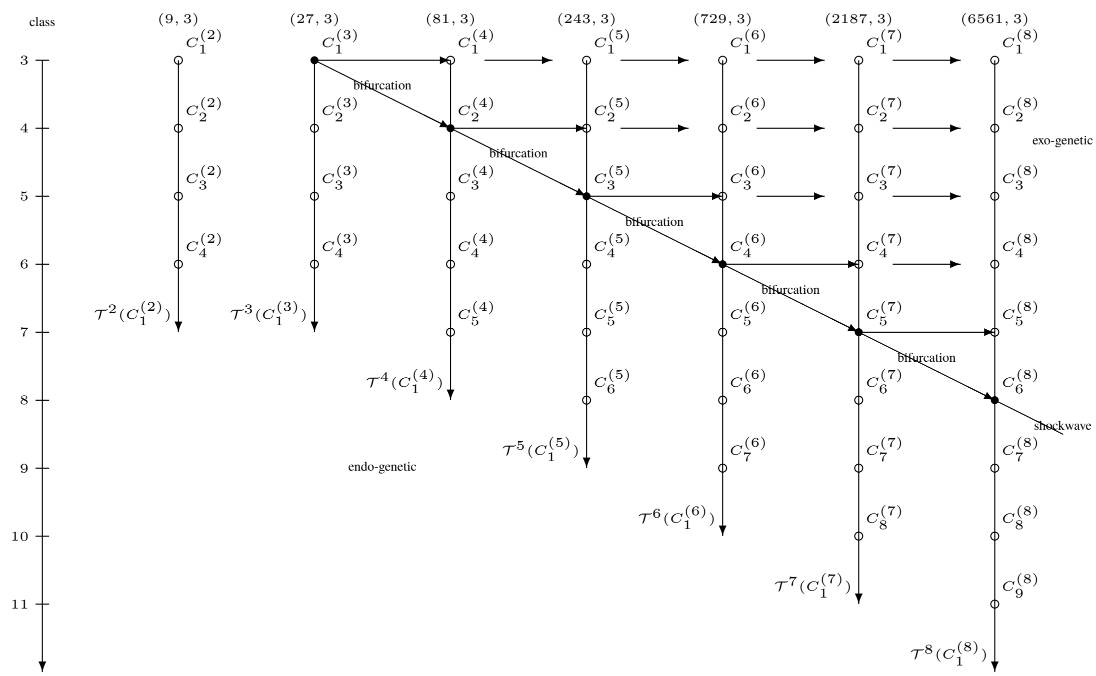
$$\pi_p(C_i^{(e)}) = C_{i-1}^{(e)}, \quad C_i^{(e)} = C_{i-1}^{(e)} - \#1;1, \tag{8}$$

  for $e \ge 3,\ i \ge e-1$*, i.e.* $\mathrm{cl}(C_i^{(e)}) > e$*.*

Remark 1 Formula (8) is the *well-known old law* for the construction of the mainline of coclass trees with elementary commutator quotient $C_3 \times C_3$. Formulas (7) and

(6) constitute the *new deterministic laws* in the finite region on and behind the shockwave, in the case of non-elementary commutator quotients $C_{3^e} \times C_3$, $e \geq 4$. The statements are illuminated graphically in Figure 1.

Fig. 1 Mainlines of CF-coclass trees and their various mechanisms of propagation



In Figure 1, the *nilpotency class* is selected as the unifying invariant on the left hand scale, since all coclass trees start at a root of class $\mathrm{cl}(C_1^{(e)}) = 3$, for $e \geq 2$. The trees are drawn for $e \leq 8$. The mainline of the leftmost coclass tree $\mathcal{T}^2(C_1^{(2)})$ with $C_1^{(2)} = \langle 243, 17\rangle$ (Ascione's CF-group A) is actually not involved in the propagation, since it is completely regular and endo-genetic: see Figure 2.

Exceptionally, the arrows of directed edges are drawn in *reverse orientation*, in order to point out the ostensive direction of bifurcation and propagation (irregular exo-genetic propagation in *horizontal* direction, singular exo-genetic propagation in *diagonal* direction, and regular endo-genetic propagation in *vertical* direction).

Figure 1 impressively shows that the root $C_1^{(3)}$ of the coclass tree $\mathcal{T}^3(C_1^{(3)})$ is a common p-ancestor of all mainline vertices of all CF-coclass trees $\mathcal{T}^e(C_1^{(e)})$, $e \geq 3$, under investigation. It is clear that $C_1^{(3)}$ is infinitely capable (root of a coclass tree), but the aforementioned fact emphasizes that $C_1^{(3)} = \langle 729, 7\rangle$ has the remarkable property of being *infinitely capable of higher order*.

Proof (Theorem 1) Let $G = \langle x, y\rangle$ be a two-generated finite 3-group. Then we denote the main commutator by $s_2 = [y, x]$ and higher commutators by $\forall_{j=3}^{c+1}\ s_j = [s_{j-1}, x]$, $t_j = [s_{j-1}, y]$. If the commutator quotient is bicyclic $G/G' \simeq C_{3^e} \times C_3$ with one non-elementary component having logarithmic exponent $e \ge 2$, then we assume $w^3 \in G'$ for $w = x^{3^{e-1}}$ and $y^3 \in G'$.

All mainline vertices involved in Theorem 1 possess a parametrized pc-presentation

$$\begin{aligned} C_i^{(e)} = \langle x, y \mid x^{3^{e-1}} = w,\ w^3 = 1,\ y^3 = 1, \\ (\forall_{j=2}^{c-3}\ s_j^3 = s_{j+2}^2 s_{j+3}),\ s_{c-2}^3 = s_c^2,\ s_{c-1}^3 = s_c^3 = 1, \\ (\forall_{j=3}^{c}\ t_j = s_j),\ t_{c+1} = s_{c+1} = 1\rangle = C_{c-2}^{(e)} \end{aligned} \tag{9}$$

with two parameters, logarithmic exponent $e \ge 2$, and nilpotency class $c = \mathrm{cl}(C_i^{(e)}) = i + 2 \ge 3$.

Recall that only the *commutator structure* enters the recursive definition of the descending central series $\gamma_1(G) = G$, and $\gamma_i(G) = [\gamma_{i-1}(G), G]$, for $i \ge 2$, but also the *power structure* is included in the lower exponent p-central series $P_0(G) = G$, and $P_i(G) = P_{i-1}(G)^3 \cdot [P_{i-1}(G), G]$, for $i \ge 1$.

Generally, for a descendant D, the parent is $A = \pi(D) = D/\gamma_c(D)$, and $A_p = \pi_p(D) = D/P_{c_p-1}(D)$ is the p-parent, where $c = \mathrm{cl}(D)$ is the class, and $c_p = \mathrm{cl}_p(D)$ is the p-class. Now we put $D := C_i^{(e)}$ and consider *three situations*.

1. *Behind* the shockwave: If $c < e$, i.e. $i+2 < e$ resp. $i \le e-3$, then $\gamma_c(D) = \langle s_c\rangle$ and $P_{c_p-1}(D) = \langle w\rangle$. Consequently, by forming the quotient we obtain $s_c = 1$ in $A = \pi(D)$ but w persists, that is $A = C_{i-1}^{(e)}$, if $i \ge 2$. However, in $A_p = \pi_p(D)$, we get $w = 1$ but s_c persists, that is $A_p = C_i^{(e-1)}$, provided that $e \ge 4$ (for $e \le 3$, the condition $3 \le c < e$ cannot occur).

2. *On* the shockwave: If $c = e$, i.e. $i + 2 = e$ resp. $i = e - 2$, then $\gamma_c(D) = \langle s_c\rangle$ and $P_{c_p-1}(D) = \langle s_c, w\rangle$ is bicyclic. Thus, by factorization we have $s_c = 1$ but w persists in A, that is $A = C_{i-1}^{(e)}$, if $e \ge 4$. However, in A_p both, $s_c = 1$ and $w = 1$, become trivial, whence $A_p = C_{i-1}^{(e-1)}$ with step size $s = 2$ reveals a bifurcation, provided that $e \ge 4$ and thus $i = e - 2 \ge 2$.

3. *Ahead of* the shockwave: If $c > e$, i.e. $i + 2 > e$ resp. $i \ge e - 1$, then $c_p = c$ and $\gamma_c(D) = P_{c_p-1}(D) = \langle s_c\rangle$. By forming the quotient we get $s_c = 1$ in $A = A_p$ but w persists, that is $A = A_p = C_{i-1}^{(e)}$, since $i \ge 3 - 1 = 2$ for each $e \ge 3$.

Strictly speaking, the preceding considerations only prove that $C_i^{(e)} = C_i^{(e-1)} - \#1; k$, resp. $C_i^{(e)} = C_{i-1}^{(e-1)} - \#2; \ell$, resp. $C_i^{(e)} = C_{i-1}^{(e)} - \#1; m$, with positive integers k, ℓ, m, but the actual implementation in Magma [18] yields $k = \ell = m = 1$ for vertices on the mainline. □

4.2 Vertices Remote from the Mainline with Depth 1

Concerning vertices V on the coclass trees $\mathcal{T}^e(C_1^{(e)})$, $e \ge 3$, which are remote from the mainline, we restrict ourselves to those with depth $\mathrm{dp}(V) = 1$ and omit the investigation of others with depth $2 \le \mathrm{dp}(V) \le 3$. Let $C_{i,j}^{(e)}$, $i, j \ge 2$, be an *offside* immediate descendant of a mainline vertex $C_{i-1}^{(e)}$, and let ζ be its center.

Theorem 2 ***(Construction)*** *The vertices $C_{i,j}^{(e)}$ remote from the mainline of the coclass tree $\mathcal{T}^e(C_1^{(e)})$ can be* ***constructed*** *recursively, according to four laws in dependence on nilpotency class and center ζ,*

- *by* ***irregular exo****-genetic propagation (behind the shockwave,* ***with stable type****)*
$$C_{i,j}^{(e)} = C_{i,j}^{(e-1)} - \#1;1, \tag{10}$$
for ζ bicyclic, $e \ge 5$, $2 \le i \le e-3$, i.e. $\mathrm{cl}(C_{i,j}^{(e)}) < e$,
- *by* ***singular exo****-genetic propagation (bifurcation on the shockwave)*
$$C_{i,j}^{(e)} = C_{i-1}^{(e-1)} - \#2;\ell, \tag{11}$$
for ζ bicyclic, $e \ge 4$, $i = e-2$, i.e. $\mathrm{cl}(C_{i,j}^{(e)}) = e$,
- *by* ***regular endo****-genetic propagation (ahead of the shockwave)*
$$C_{i,j}^{(e)} = C_{i-1}^{(e)} - \#1;m, \tag{12}$$
for ζ bicyclic, $e \ge 3$, $i \ge e-1$, i.e. $\mathrm{cl}(C_{i,j}^{(e)}) > e$,
- *by* ***permanent regular endo****-genetic propagation (independent of the shockwave)*
$$C_{i,j}^{(e)} = C_{i-1}^{(e)} - \#1;q, \tag{13}$$
for ζ cyclic, $e \ge 3$, $i \ge 2$.

Proof For each *periodic sequence* (also called *coclass family*), the vertices V have a parametrized pc-presentation with two parameters e and c. By the **mainline principle**, the generating commutator of the last non-trivial lower central $\gamma_c(V) = \langle s_c \rangle$ does not enter the relations for the mainline, but enters at least one typical relation, in **boldface** font, for each vertex off mainline. The branches of the coclass trees under investigation are periodic with length 2. On every branch, there is a unique mainline vertex M of type $\mathrm{a}.1$, $\varkappa(M) = (000;0)$. We recall its pc-presentation:

$$\begin{aligned} M=\langle x,y \mid x^{3^{e-1}}&=w,\ w^3=1,\ y^3=1,\\ &(\forall_{j=2}^{c-3}\ s_j^3=s_{j+2}^2 s_{j+3}),\ s_{c-2}^3=s_c^2,\ s_{c-1}^3=s_c^3=1,\\ &(\forall_{j=3}^{c}\ t_j=s_j),\ t_{c+1}=s_{c+1}=1\rangle=C_{c-2}^{(e)}=C_{c-2,1}^{(e)} \end{aligned} \tag{14}$$

Furthermore, there is a unique leaf V of type $\mathrm{b}.16$, $\varkappa(V)\sim(004;0)$:

$$\begin{aligned} V=\langle x,y \mid x^{3^{e-1}}&=w,\ w^3=1,\ \mathbf{y^3=s_c},\\ &(\forall_{j=2}^{c-3}\ s_j^3=s_{j+2}^2 s_{j+3}),\ s_{c-2}^3=s_c^2,\ s_{c-1}^3=s_c^3=1,\\ &(\forall_{j=3}^{c}\ t_j=s_j),\ t_{c+1}=s_{c+1}=1\rangle=C_{c-2,2}^{(e)} \end{aligned} \tag{15}$$

On every branch, there is a unique root V of type $\mathrm{a}.1$, $\varkappa(V)=(000;0)$, of a *twig* which goes down to depth 3 (we devote our attention to the root alone and abstain from its descendants):

$$\begin{aligned} V=\langle x,y \mid x^{3^{e-1}}&=w,\ w^3=1,\ y^3=1,\\ &(\forall_{j=2}^{c-3}\ s_j^3=s_{j+2}^2 s_{j+3}),\ s_{c-2}^3=s_c^2,\ s_{c-1}^3=s_c^3=1,\\ &\mathbf{t_3=s_3s_c},\ (\forall_{j=4}^{c}\ t_j=s_j),\ t_{c+1}=s_{c+1}=1\rangle=C_{c-2,3}^{(e)} \end{aligned} \tag{16}$$

On every branch, there are two leaves V of type $\mathrm{a}.1$, $\varkappa(V)=(000;0)$, with *bicyclic* center ζ and exponent $1\le n\le 2$:

$$\begin{aligned} V=\langle x,y \mid x^{3^{e-1}}&=w,\ w^3=1,\ \mathbf{y^3=s_c^n},\\ &(\forall_{j=2}^{c-3}\ s_j^3=s_{j+2}^2 s_{j+3}),\ s_{c-2}^3=s_c^2,\ s_{c-1}^3=s_c^3=1,\\ &\mathbf{t_3=s_3s_c},\ (\forall_{j=4}^{c}\ t_j=s_j),\ t_{c+1}=s_{c+1}=1\rangle\\ &=C_{c-2,4}^{(e)} \text{ and } C_{c-2,5}^{(e)} \end{aligned} \tag{17}$$

On odd branches, we have a single leaf, on even branches, we have two leaves, V of type $\mathrm{b}.3$, $\varkappa(V)\sim(001;0)$, with *cyclic* center ζ and exponent $n=1$, respectively $1\le n\le 2$:

$$\begin{aligned} V=\langle x,y \mid x^{3^{e-1}}&=w,\ \mathbf{w^3=s_c^n},\ y^3=1,\\ &(\forall_{j=2}^{c-3}\ s_j^3=s_{j+2}^2 s_{j+3}),\ s_{c-2}^3=s_c^2,\ s_{c-1}^3=s_c^3=1,\\ &(\forall_{j=3}^{c}\ t_j=s_j),\ t_{c+1}=s_{c+1}=1\rangle\\ &=C_{c-2,6}^{(e)} \text{ (and } C_{c-2,8}^{(e)} \text{ on even branches)} \end{aligned} \tag{18}$$

On odd branches, we have a single leaf, on even branches, we have two leaves, V of type $\mathrm{a}.1$, $\varkappa(V)=(000;0)$, with *cyclic* center ζ and exponent $n=1$, respectively $1\le n\le 2$:

$$\begin{aligned} V = \langle x, y \mid x^{3^{e-1}} = w,\ \mathbf{w^3} = \mathbf{s_c^n},\ y^3 = 1, \\ (\forall_{j=2}^{c-3}\ s_j^3 = s_{j+2}^2 s_{j+3}),\ s_{c-2}^3 = s_c^2,\ s_{c-1}^3 = s_c^3 = 1, \\ \mathbf{t_3} = \mathbf{s_3 s_c},\ (\forall_{j=4}^{c}\ t_j = s_j),\ t_{c+1} = s_{c+1} = 1\rangle \\ = C_{c-2,7}^{(e)} \text{ (and } C_{c-2,9}^{(e)} \text{ on even branches)} \end{aligned} \tag{19}$$

Similarly as in the proof of Theorem 1, for a descendant D, the parent is $A = \pi(D) = D/\gamma_c(D)$, and the p-parent is $A_p = \pi_p(D) = D/P_{c_p-1}(D)$, where $c = \mathrm{cl}(D)$ is the class, and $c_p = \mathrm{cl}_p(D)$ is the p-class. Now we put $D := C_{i,j}^{(e)}$ and consider *four situations*. For the first three items, let D be a vertex with *bicyclic center*, and thus with one of the presentations (15), (16) or (17).

1. *Behind* the shockwave: If $c < e$, i.e. $i+2 < e$ resp. $i \le e-3$, then $\gamma_c(D) = \langle s_c\rangle$ and $P_{c_p-1}(D) = \langle w\rangle$. Consequently, we obtain $s_c = 1$ in $A = \pi(D)$ but w persists, that is $A = C_{i-1}^{(e)}$, if $i \ge 2$. However, in $A_p = \pi_p(D)$, we get $w = 1$ but s_c (and the distinguished relation) persists, that is $A_p = C_{i,j}^{(e-1)}$, same type, provided that $e \ge 5$ (for $e \le 4$, condition $4 \le c < e$ cannot occur).

2. *On* the shockwave: If $c = e$, i.e. $i + 2 = e$ resp. $i = e - 2$, then $\gamma_c(D) = \langle s_c\rangle$ and $P_{c_p-1}(D) = \langle s_c, w\rangle$ is bicyclic. Thus, we have $s_c = 1$ but w persists in A, that is $A = C_{i-1}^{(e)}$, if $e \ge 4$. However, in A_p both, $s_c = 1$ and $w = 1$, become trivial, whence $A_p = C_{i-1}^{(e-1)}$ with step size $s = 2$ reveals a bifurcation, provided that $e \ge 4$ and thus $i = e - 2 \ge 2$.

3. *Ahead of* the shockwave: If $c > e$, i.e. $i + 2 > e$ resp. $i \ge e - 1$, then $c_p = c$ and $\gamma_c(D) = P_{c_p-1}(D) = \langle s_c\rangle$. So we get $s_c = 1$ (the distinguished relation degenerates) in $A = A_p$ but w persists, i.e. we get a mainline vertex $A = A_p = C_{i-1}^{(e)}$, since $i \ge 3 - 1 = 2$ for each $e \ge 3$.

4. Let D be a vertex with *cyclic center*, and thus with one of the presentations (18) or (19). Although the p-class c_p may be bigger than the class c of D, nevertheless, the last non-trivial lower central and lower p-central coincide $\gamma_c(D) = P_{c_p-1}(D) = \langle s_c\rangle$, due to the *exceptional relation* $\mathbf{w^3} = \mathbf{s_c^n}$. So we get $s_c = 1$ in $A = A_p$ and $w^3 = 1$ becomes trivial, that is, the propagation is always regular and endo-genetic with coinciding parent and p-parent $A = A_p = C_{i-1}^{(e)}$ a mainline vertex, where $i + 2 = c \ge 4$, i.e. $i - 1 = c - 3 \ge 1$.

Strictly speaking, the preceding considerations only prove that $C_{i,j}^{(e)} = C_{i,j}^{(e-1)} - \#1; k$, resp. $C_{i,j}^{(e)} = C_{i-1}^{(e-1)} - \#2; \ell$, resp. $C_{i,j}^{(e)} = C_{i-1}^{(e)} - \#1; m$, resp. $C_{i,j}^{(e)} = C_{i-1}^{(e)} - \#1; q$, with positive integers k, ℓ, m, q, but the actual implementation in Magma [18] yields $k = 1$, and $\ell, m, q \ge 2$. For exact values of ℓ, m, q see the proofs of Theorems 4–5. □

Remark 2 It should be pointed out that the descendant vertices V of each root $C_1^{(e)}$ share further invariants with the root, aside from the rank distribution $\varrho(V)$. They have closely related transfer kernel types $\varkappa(V)$ with three identical components (the *stabilization*) and a single varying component (the *polarization*). For all CF trees in this paper, the polarization is located at the *third* component, and thus distinct from the *puncture*, which is the fourth component, by convention.

5 Laws for Coclass Trees of BCF-Groups

Again, we separate our main statements into three parts: uniqueness, invariants, and construction.

Proposition 3 ***(Uniqueness)*** *For each log exponent* $e \geq 2$*, there exists a unique coclass tree* $\mathcal{T}^{e+1}(B_1^{(e+1)}) \ni V$ *with fixed coclass* $\mathrm{cc}(V) = e+1$*, fixed commutator quotient* $V/V' \simeq C_{3^e} \times C_3$*, and fixed rank distribution* $\varrho(V) \sim (2,2,3;3)$*. Its mainline* $(B_i^{(e+1)})_{i\geq 1}$ *is of type* $\mathrm{d}.10$*,* $\varkappa(B_i^{(e+1)}) \sim (1,1,0;2)$*. The tree contains metabelian and non-metabelian* BCF*-groups. The branches are of depth* 4*. (See Figures 5–7 for the depth-pruned metabelian skeleton, when* $2 \leq e \leq 4$*.)*

Proof According to the proof of Proposition 1, there are only two coclass trees $\mathcal{T}^r(R_j^{(r)})$ with rank distribution $\varrho(V) \sim (2,2,3;3)$, for each $e \geq 2$, $e \leq r \leq e+1$, a BCF-tree of type $\mathrm{d}.10$ and a CF-tree of type $\mathrm{a}.1$. The former is the *unique* tree with root $R_j^{(e+1)} = B_1^{(e+1)}$, recursively determined by the CF-group $C_1^{(3)} = \langle 729, 7\rangle$ and the BCF-group $B_1^{(3)} = \langle 729, 13\rangle$. Its depth-pruned metabelian branches are periodic of length 2 without pre-period, and all of its vertices are BCF-groups, since the vertices of the first two branches are BCF-groups. □

Proposition 4 ***(Invariants)*** *For* $e \geq 2$*, invariants of vertices on the mainline* $(B_i^{(e+1)})_{i\geq 1}$ *of the coclass tree* $\mathcal{T}^{e+1}(B_1^{(e+1)})$ *are given as follows:*

$$
\begin{aligned}
\textit{logarithmic order } \mathrm{lo}(B_i^{(e+1)}) &= e+i+3, \textit{ for } i \geq 1,\\
\textit{nilpotency class } \mathrm{cl}(B_i^{(e+1)}) &= i+2, \textit{ for } i \geq 1,\\
p\textit{-class } \mathrm{cl}_p(B_i^{(e+1)}) &= \begin{cases} i+2 & \textit{if } i > e-1,\\ e+1 & \textit{if } i \leq e-1,\end{cases}\\
p\textit{-coclass } \mathrm{cc}_p(B_i^{(e+1)}) &= \begin{cases} e+1 & \textit{if } i > e-1,\\ i+2 & \textit{if } i \leq e-1.\end{cases}
\end{aligned}
\tag{20}
$$

Proof Proposition 4 remains true when the mainline vertex $B_i^{(e+1)}$ is replaced by any proper descendant vertex $B_{i,j}^{(e+1)}$ with $i \geq 2$. All coclass trees under investigation start at a root of class $\mathrm{cl}(B_1^{(e+1)}) = 3 = 1+2$, for each $e \geq 2$. Thus, proper descendants possess nilpotency class $\mathrm{cl}(B_{i,j}^{(e+1)}) = i+2 \geq 4$. By definition, all vertices V of the coclass tree $\mathcal{T}^{e+1}(B_1^{(e+1)})$ share the common coclass $\mathrm{cc}(V) = e+1$. Consequently, the logarithmic order is the sum $\mathrm{lo}(B_{i,j}^{(e+1)}) = \mathrm{cl}(B_{i,j}^{(e+1)}) + \mathrm{cc}(B_{i,j}^{(e+1)}) = i+2+e+1$. Finally, the *power structure* of all finite 3-groups G

with commutator quotient $G/G' \simeq C_{3^e} \times C_3$ is responsible for the constant p-class $\mathrm{cl}_p(B_{i,j}^{(e+1)}) = e+1$, independently of the class $\mathrm{cl}(B_{i,j}^{(e+1)}) = i+2 \le e+1$, in the finite region on and behind the shockwave. □

Concerning vertices V on the coclass trees $\mathcal{T}^{e+1}(B_1^{(e+1)})$, $e \ge 2$, which are remote from the mainline, we restrict ourselves to the metabelian with depth $\mathrm{dp}(V) = 1$, and we omit the investigation of others with depth $2 \le \mathrm{dp}(V) \le 4$.

Let $B_{i,j}^{(e+1)}$ with $i \ge 1$ be any metabelian vertex *on or remote from* the mainline $(B_i^{(e+1)})_{i\ge 1}$.

Theorem 3 ***(Construction)*** *The vertices $B_{i,j}^{(e+1)}$ on and remote from the mainline of the coclass tree $\mathcal{T}^{e+1}(B_1^{(e+1)})$ can be* ***constructed*** *recursively, according to three laws in dependence on the nilpotency class,*

- *by* ***irregular endo****-genetic propagation (behind the shockwave,* ***with type change****)*
$$B_{i,j}^{(e+1)} = C_{i,\ell}^{(e)} - \#1; k, \tag{21}$$
for $e \ge 3$, $1 \le i \le e-2$, i.e. $\mathrm{cl}(B_{i,j}^{(e+1)}) < e+1$, where the types d.10*,* B.2*,* D.10*,* C.4*,* D.5*, of $B_{i,j}^{(e+1)}$ correspond to the types* a.1 *mainline,* a.1 *twig root,* b.16*,* a.1 *bicyclic center,* a.1 *bicyclic center, of $C_{i,\ell}^{(e)}$, respectively,*
- *by* ***singular endo****-genetic propagation (bifurcation on the shockwave)*
$$B_{i,j}^{(e+1)} = C_{i-1}^{(e)} - \#2; m, \tag{22}$$
for $e \ge 3$, $i = e-1$, i.e. $\mathrm{cl}(B_{i,j}^{(e+1)}) = e+1$,
- *by* ***regular endo****-genetic propagation (ahead of the shockwave)*
$$B_{i,j}^{(e+1)} = B_{i-1}^{(e+1)} - \#1; q, \tag{23}$$
for $e \ge 2$, $i \ge e$, i.e. $\mathrm{cl}(B_{i,j}^{(e+1)}) > e+1$.

Proof For each *periodic sequence* (or *coclass family*), the vertices V have a parametrized pc-presentation with two parameters e and c. According to the **mainline principle**, the generating commutator of the last non-trivial lower central $\gamma_c(V) = \langle s_c \rangle$ does not enter the relations for the mainline, but enters at least one typical relation, in **boldface** font, for each vertex off mainline. Branches of coclass trees under investigation are periodic with length 2. On every branch, there is a unique mainline vertex M of type d.10, $\varkappa(M) \sim (110; 2)$. Its pc-presentation is given by:

$$
\begin{aligned}
M = \langle x, y \mid x^{3^e} = w,\ w^3 = 1,\ y^3 = 1,\\
&(\forall_{j=2}^{c-3}\ s_j^3 = s_{j+2}^2 s_{j+3}),\ s_{c-2}^3 = s_c^2,\ s_{c-1}^3 = s_c^3 = 1,\\
&t_3 = s_3 w,\ (\forall_{j=4}^{c}\ t_j = s_j),\ t_{c+1} = s_{c+1} = 1\rangle\\
&= B_{c-2}^{(e+1)} = B_{c-2,1}^{(e+1)}
\end{aligned} \tag{24}
$$

On odd branches, we have a single vertex, on even branches, we have two vertices, V of type D.10, $\varkappa(V) \sim (114; 2)$, with exponent $n = 1$, respectively $1 \le n \le 2$:

$$
\begin{aligned}
V = \langle x, y \mid x^{3^e} = w,\ w^3 = 1,\ \mathbf{y^3 = s_c^n},\\
&(\forall_{j=2}^{c-3}\ s_j^3 = s_{j+2}^2 s_{j+3}),\ s_{c-2}^3 = s_c^2,\ s_{c-1}^3 = s_c^3 = 1,\\
&t_3 = s_3 w,\ (\forall_{j=4}^{c}\ t_j = s_j),\ t_{c+1} = s_{c+1} = 1\rangle\\
&= B_{c-2,h}^{(e+1)},\ h = 2 \text{ (or } h = 2, 3)
\end{aligned} \tag{25}
$$

On odd branches, we have a single root, on even branches, we have two roots, V of type B.2, $\varkappa(V) \sim (111; 2)$, of a twig, with exponent $n = 1$, respectively $1 \le n \le 2$:

$$
\begin{aligned}
V = \langle x, y \mid x^{3^e} = w,\ w^3 = 1,\ y^3 = 1,\\
&(\forall_{j=2}^{c-3}\ s_j^3 = s_{j+2}^2 s_{j+3}),\ s_{c-2}^3 = s_c^2,\ s_{c-1}^3 = s_c^3 = 1,\\
&\mathbf{t_3 = s_3 s_c^n w},\ (\forall_{j=4}^{c}\ t_j = s_j),\ t_{c+1} = s_{c+1} = 1\rangle\\
&= B_{c-2,h}^{(e+1)},\ h = 3 \text{ (or } h = 4, 7)
\end{aligned} \tag{26}
$$

On odd branches, we have a single vertex, on even branches, we have two vertices, V of type C.4, $\varkappa(V) = (112; 2)$, with exponent $n = 1$, respectively $1 \le n \le 2$:

$$
\begin{aligned}
V = \langle x, y \mid x^{3^e} = w,\ w^3 = 1,\ \mathbf{y^3 = s_c^n},\\
&(\forall_{j=2}^{c-3}\ s_j^3 = s_{j+2}^2 s_{j+3}),\ s_{c-2}^3 = s_c^2,\ s_{c-1}^3 = s_c^3 = 1,\\
&\mathbf{t_3 = s_3 s_c^n w},\ (\forall_{j=4}^{c}\ t_j = s_j),\ t_{c+1} = s_{c+1} = 1\rangle\\
&= B_{c-2,h}^{(e+1)},\ h = 4 \text{ (or } h = 5, 9)
\end{aligned} \tag{27}
$$

On odd branches, we have a single vertex, on even branches, we have two vertices, V of type D.5, $\varkappa(V) = (113; 2)$, with exponent $n = 1$, respectively $1 \le n \le 2$:

$$
\begin{aligned}
V = \langle x, y \mid x^{3^e} = w,\ w^3 = 1,\ \mathbf{y^3 = s_c^n},\\
&(\forall_{j=2}^{c-3}\ s_j^3 = s_{j+2}^2 s_{j+3}),\ s_{c-2}^3 = s_c^2,\ s_{c-1}^3 = s_c^3 = 1,\\
&\mathbf{t_3 = s_3 s_c^{3-n} w},\ (\forall_{j=4}^{c}\ t_j = s_j),\ t_{c+1} = s_{c+1} = 1\rangle\\
&= B_{c-2,h}^{(e+1)},\ h = 5 \text{ (or } h = 6, 8)
\end{aligned} \tag{28}
$$

Similarly as in the proof of Theorem 2, for a descendant D, the parent is $A = \pi(D) = D/\gamma_c(D)$, and $A_p = \pi_p(D) = D/P_{c_p-1}(D)$ is the p-parent, where

$c = \mathrm{cl}(D)$ is the class, and $c_p = \mathrm{cl}_p(D)$ is the p-class. Now we put $D := B^{(e+1)}_{i,j}$ and consider *three situations*.

1. *Behind* the shockwave: If $c < e+1$, i.e. $i+2 < e+1$ resp. $i \le e-2$, then $\gamma_c(D) = \langle s_c, t_c\rangle$, if $i = 1$, $\gamma_c(D) = \langle s_c\rangle$, if $i \ge 2$, and $P_{c_p-1}(D) = \langle w\rangle$. Consequently, if $i \ge 2$, we obtain $s_c = 1$ in $A = \pi(D)$ but w persists, all Formulas (24), (25), (26), (27), (28) degenerate to (24), that is $A = B^{(e+1)}_{i-1}$, if $i \ge 2$. However, in $A_p = \pi_p(D)$, we get $w = 1$ but s_c (and the distinguished relation) persists, Formula (24), resp. (25), (26), (27), and (28), becomes Formula (14), resp. (15), (16), (17), and the same (17), i.e. $A_p = C^{(e)}_{i,\ell}$, provided $e \ge 4$ (for $e \le 3$, condition $4 \le c < e+1$ cannot occur) or $e = 3$ and $c = 3$.

2. *On* the shockwave: If $c = e+1$, i.e. $i+2 = e+1$ resp. $i = e-1$, then $\gamma_c(D) = \langle s_c\rangle$ and $P_{c_p-1}(D) = \langle s_c, w\rangle$ is bicyclic. Thus, we have $s_c = 1$ but w persists in A, that is $A = B^{(e+1)}_{i-1}$, if $e \ge 3$. However, in A_p both, $s_c = 1$ and $w = 1$, become trivial, all Formulas (24), (25), (26), (27), (28) degenerate to (14), whence $A_p = C^{(e)}_{i-1}$ with step size $s = 2$ reveals a bifurcation, provided that $e \ge 3$ and thus $i = e-1 \ge 2$.

3. *Ahead of* the shockwave: If $c > e+1$, i.e. $i+2 > e+1$ resp. $i \ge e$, then $c_p = c$ and $\gamma_c(D) = P_{c_p-1}(D) = \langle s_c\rangle$. So we get $s_c = 1$ in $A = A_p$ but w persists, all Formulas (24), (25), (26), (27), (28) degenerate to (24), that is a mainline vertex $A = A_p = B^{(e+1)}_{i-1}$, since $i \ge 2$ for each $e \ge 2$.

In the preceding developments, we have proved that $B^{(e+1)}_{i,j} = C^{(e)}_{i,\ell} - \#1; k$, resp. $B^{(e+1)}_{i,j} = C^{(e)}_{i-1} - \#2; m$, resp. $B^{(e+1)}_{i,j} = B^{(e+1)}_{i-1} - \#1; q$, with positive integers k, m, q. For exact values of j, ℓ, k, m, q see the proofs of Theorems 4–5. □

6 Construction algorithm

Since the lack of a p-capable root prohibits the recursive construction of a BCF-coclass tree $\mathcal{T}^{e^*+1}(B^{(e^*+1)}_{1,1})$ with logarithmic exponent $e^* \ge 3$ by means of iterated applications of the p-group generation algorithm [13, 1, 2], the *new deterministic laws* in Theorems 1, 2, and 3 must be employed in order to design a *new algorithm* for the branches *behind and on the shockwave*. If it is sufficient to proceed to the smallest p-capable root from which the usual endo-genetic propagation is possible, then only Step 1 is required. When the *complete coclass tree* is desired, Step 2 must be executed additionally. Only BCF- but no CF-trees are useful for imaginary quadratic base fields.

Algorithm **(Construction of BCF-Tree)** Let $e^* \ge 3$ be an assigned logarithmic exponent.

- **Step 1.** Diagonal exo-genetic propagation along the main trunk with periodic bifurcations (step size $s = 2$), recursively according to Equation (7),

$$C^{(e+1)}_{e-1,1} = C^{(e)}_{e-2,1} - \#2; 1 \text{ for } e = 3, \ldots, e^* - 1,$$

starting with $C_{1,1}^{(3)} = \langle 729, 7\rangle$ and terminating with $C_{e^*-2,1}^{(e^*)}$. Then,

$$B_{e^*-1,i-5}^{(e^*+1)} = C_{e^*-2,1}^{(e^*)} - \#2; i \text{ for } i = 6, \dots, 10, \text{ respectively } 14,$$

according to Equation (22).

- **Step 2.** Horizontal exo-genetic propagation along a singlet ($h = 1$) and several quintets ($2 \le h \le e^* - 2$) of periodic chains with step size $s = 1$, recursively according to Equation (10), a singlet $C_{1,1}^{(e+1)} = C_{1,1}^{(e)} - \#1; 1$ for $e = 3, \dots, e^* - 1$, and $e^* - 3$ quintets

$$C_{h,j}^{(e+1)} = C_{h,j}^{(e)} - \#1; 1 \text{ for } e = h+2, \dots, e^* - 1,\ j = 1, \dots, 5,$$

starting with $C_{h,j}^{(h+2)} = C_{h-1,1}^{(h+1)} - \#2, j$, according to Equation (11), using $C_{h-1,1}^{(h+1)}$ from Step 1, and terminating with $C_{h,j}^{(e^*)}$. Then, $B_{1,1}^{(e^*+1)} = C_{1,1}^{(e^*)} - \#1; 2$ and

$$B_{h,i}^{(e^*+1)} = C_{h,j}^{(e^*)} - \#1; 2, \text{ respectively } - \#1; 3, \text{ for } j = 1, \dots, 5,$$

and $i = 1, \dots, 5,$ respectively 9, according to Equation (21). □

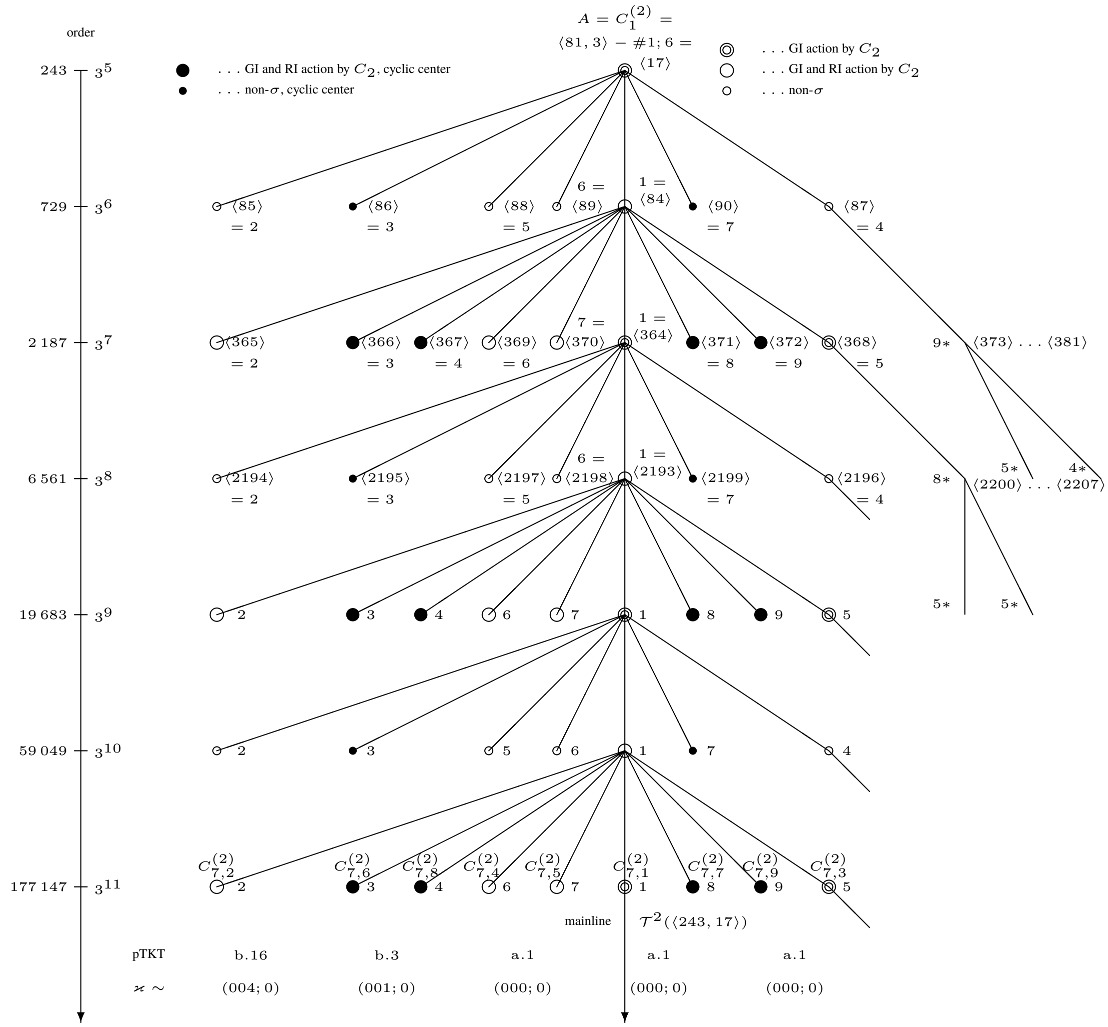
So the main navigation is performed exclusively along the skeleton of CF-groups with bicyclic center ζ. Only the trailing construction of each step concerns BCF-groups.

7 Concrete coclass trees of CF-groups

In Figure 2, the coclass tree with *Ascione's CF-group A* as its root $\langle 243, 17\rangle = \langle 81, 3\rangle - \#1; 6$ is drawn up to order 3^{11}. The branches are periodic with length 2 and naturally bounded depth 3, without artificial pruning. The infinite mainline is of type $\mathrm{a}.1$ and consists of σ-groups with GI action by C_2. The other vertices are σ-groups for even branches and non-σ groups for odd branches. Vertices with positive depth are of type $\mathrm{b}.16$ or $\mathrm{b}.3$ or $\mathrm{a}.1$. All vertices of type $\mathrm{b}.3$ and some of type $\mathrm{a}.1$ have a cyclic center $\zeta \simeq C_9$. Vertices of depth $\mathrm{dp} \in \{2, 3\}$ are exclusively of type $\mathrm{a}.1$ with cyclic center $\zeta \simeq C_3$. They are drawn for the first and second branch only. On the first branch, $\langle 373\rangle$ gives rise to $\langle 2208\rangle \dots \langle 2212\rangle$, and $\langle 378\rangle$ gives rise to $\langle 2213\rangle \dots \langle 2216\rangle$. On the second branch, $\langle 2200\rangle$ gives rise to 5 non-σ-descendants, and $\langle 2205\rangle$ gives rise to 5 descendants with generator inverting (GI) and relator inverting (RI) action by C_2.

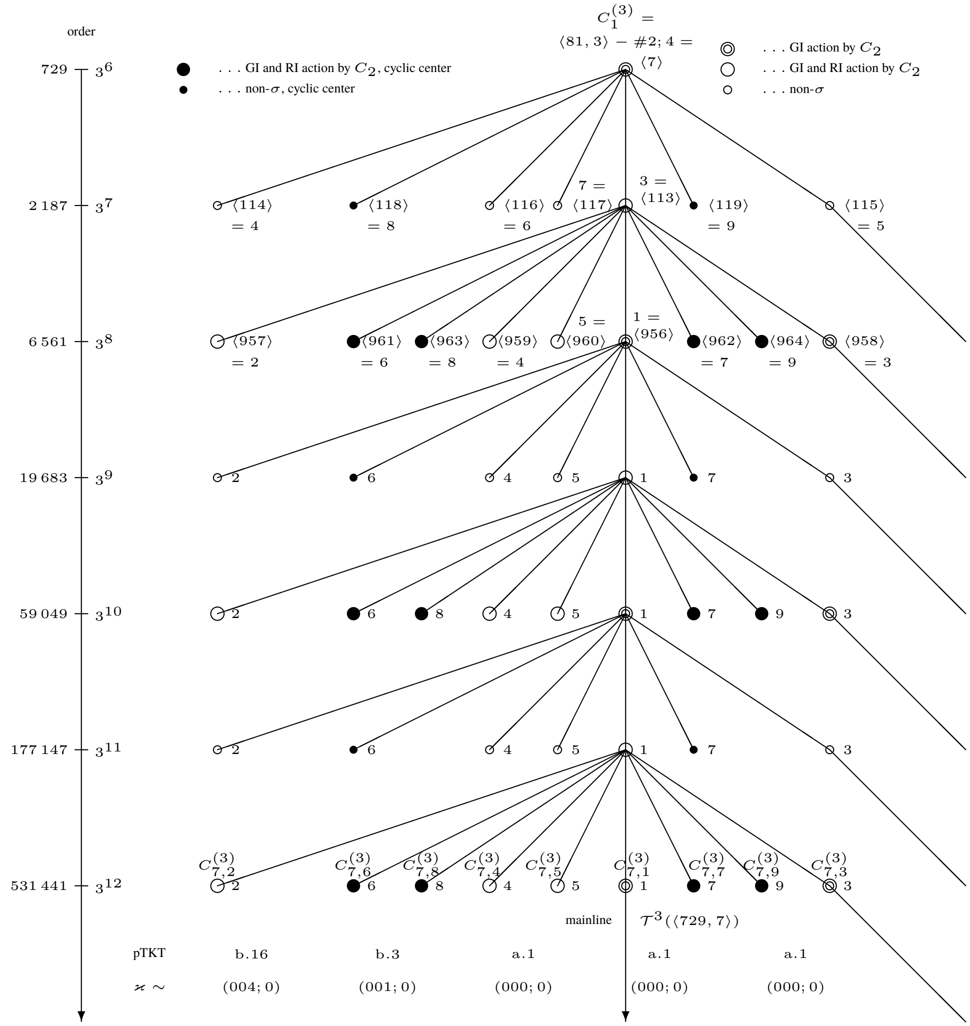
Remark 3 The purely graph theoretic structure of the coclass tree in Figure 2 was indicated in [4, Tbl. 6, p. 272] with much less details and only up to order 3^8. The root $A = \langle 243, 17\rangle$ was described ten years earlier by James [19] as a member of the first branch $\Phi_3(1)$ of Hall's isoclinism class Φ_3. See the pc-presentation for $\Phi_3(2111)e \simeq A$ in [20, p. 620]. Another pc-presentation for A can be extracted from the microfiches in [4, p. 320, folio I02].

Fig. 2 Coclass-2 CF-tree of type $a.1$, rooted in original Ascione A, with AQI (21)



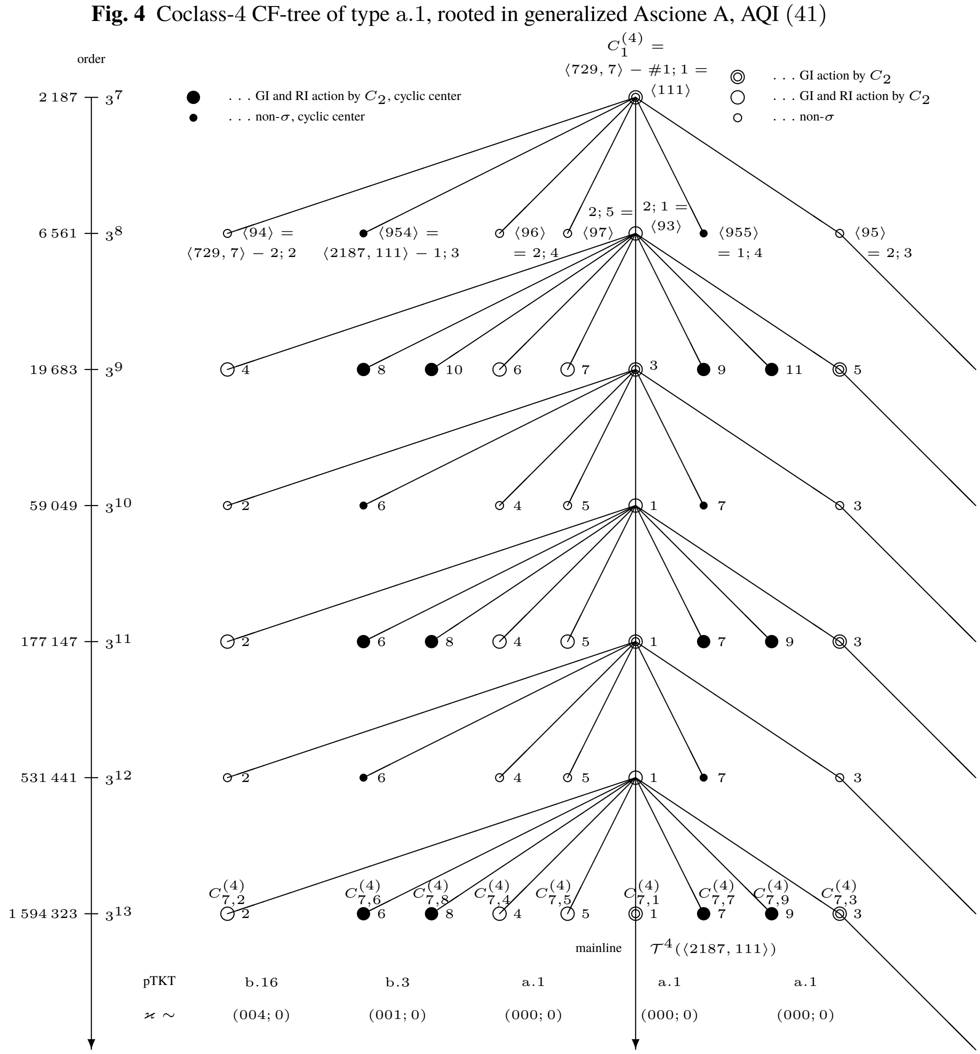
Similarly as in Figure 2, the *propagation* in Figure 3 is also completely *regular*, and purely *endo*-genetic with respect to the coclass tree $\mathcal{T}^3(C_1^{(3)})$. However, the *generalized CF-group A of Ascione*, $C_1^{(3)} \simeq \langle 729, 7\rangle$, is located *on* the shock wave, and consequently also possesses additional *exo*-genetic p-descendants with step size $s = 1$, namely $C_1^{(3)} - \#1; i$ with $i \in \{1, 2\}$. This leads to *exceptional relative identifiers* for the vertices of the first branch $\mathcal{B}(C_1^{(3)})$, beginning with $C_2^{(3)} = C_1^{(3)} - \#1; 3 \simeq \langle 2187, 113\rangle$. (The mainline vertex $C_i^{(3)}$ of all other branches with $i \geq 3$ has relative identifier $C_{i-1}^{(3)} - \#1; 1$.)

Fig. 3 Coclass-3 CF-tree of type a.1, rooted in generalized Ascione A, AQI (31)



As opposed to the Figures 2 and 3, the entire *first branch* is *exceptional* in Figure 4. The root $C_1^{(4)} \simeq \langle 2187, 111\rangle$, that is the *generalized CF-group A of Ascione*, arises by *irregular exo*-genetic propagation as p-descendant $C_1^{(3)} - \#1; 1$ with step size $s = 1$ of the periodic root $C_1^{(3)} \simeq \langle 729, 7\rangle$ of bifurcations, according to Formula (6). The vertices $C_{2,i}^{(4)} = C_1^{(3)} - \#2; i$ with $1 \le i \le 5$ and *bicyclic* center arise by *singular exo*-genetic propagation as p-descendants with step size $s = 2$ of $C_1^{(3)}$, according to Formulas (7) and (11). The vertices $C_{2,i+3}^{(4)} = C_1^{(4)} - \#1; i$ with $i \in \{3, 4\}$ and *cyclic* center, however, arise by *permanent regular endo*-genetic propagation as p-descendants with step size $s = 1$ of the root $C_1^{(4)} \simeq \langle 2187, 111\rangle$ of a *periodic chain* with step size $s = 1$, according to Formula (13). See also the Exhaustion Theorem 6.

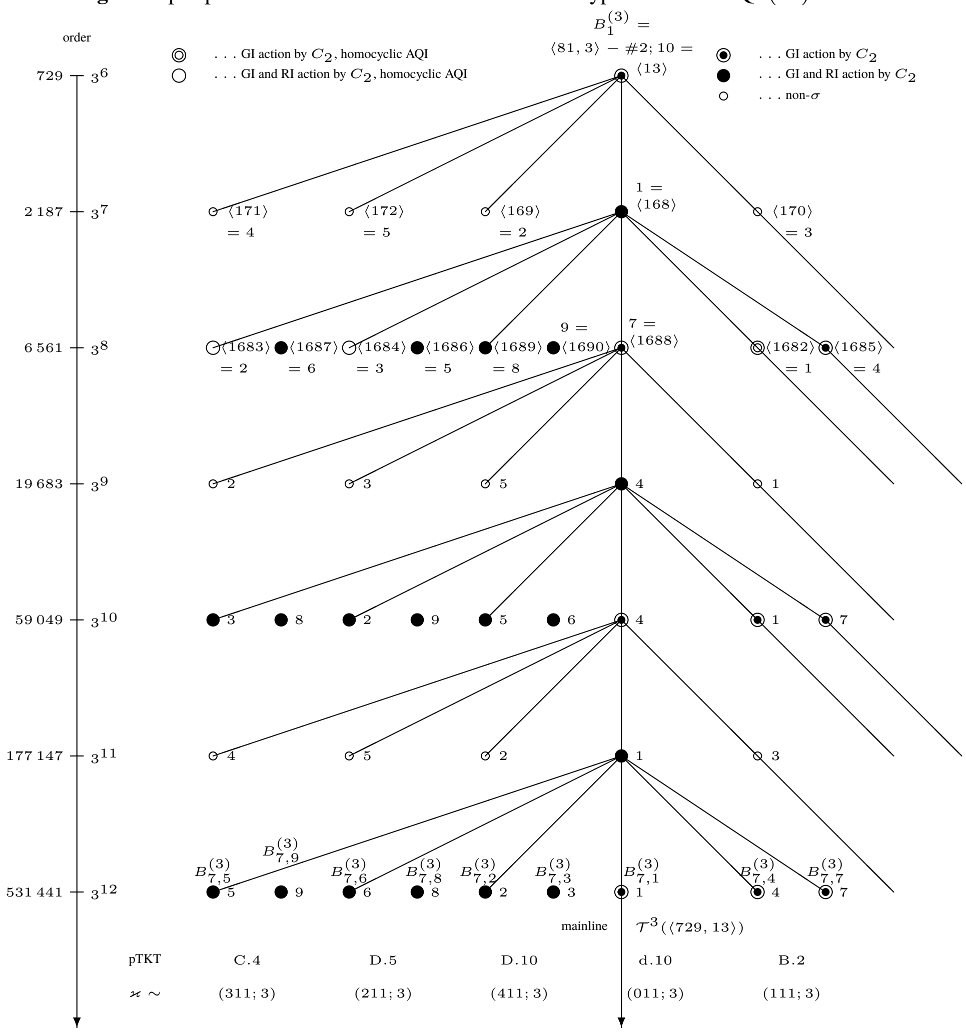
Fig. 4 Coclass-4 CF-tree of type $\mathrm{a}.1$, rooted in generalized Ascione A, AQI (41)



8 Concrete coclass trees of BCF-groups

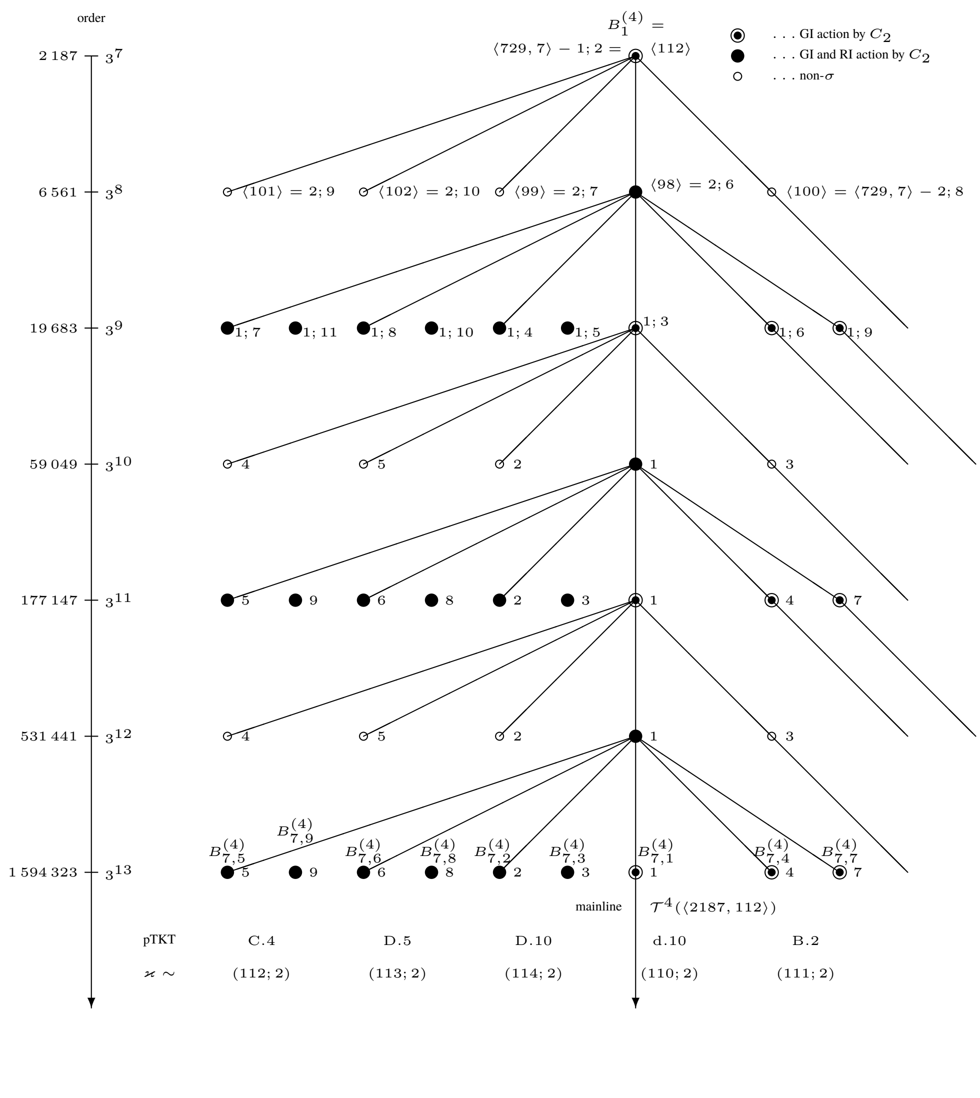
In Figure 5, the *depth-pruned metabelian skeleton* of the coclass tree with root $\langle 729, 13\rangle = \langle 81, 3\rangle - \#2; 10$ is drawn up to order 3^{12}. Without artificial pruning, the branches have naturally bounded depth 4 (not drawn). The depth-pruned metabelian branches are periodic with length 2. The infinite mainline is of type $\mathrm{d}.10$ and consists of σ-groups with GI action by C_2, every other even with RI action. The offside vertices are σ-groups for even branches and non-σ groups for odd branches. Vertices with positive depth are of type $\mathrm{D}.10$, $\mathrm{D}.5$, $\mathrm{C}.4$, $\mathrm{B}.2$ or $\mathrm{d}.10$. Metabelian vertices of depth $\mathrm{dp} \geq 2$ are exclusively of type $\mathrm{B}.2$, for instance, $\langle 2187, 170\rangle$ has 105 descendants with step size $s = 1$, whereas $\langle 6561, 1682\rangle$ and $\langle 6561, 1685\rangle$ have 58 descendants each.

Fig. 5 Depth-pruned metabelian coclass-3 BCF-tree of type $\mathrm{d}.10$ with AQI (21)



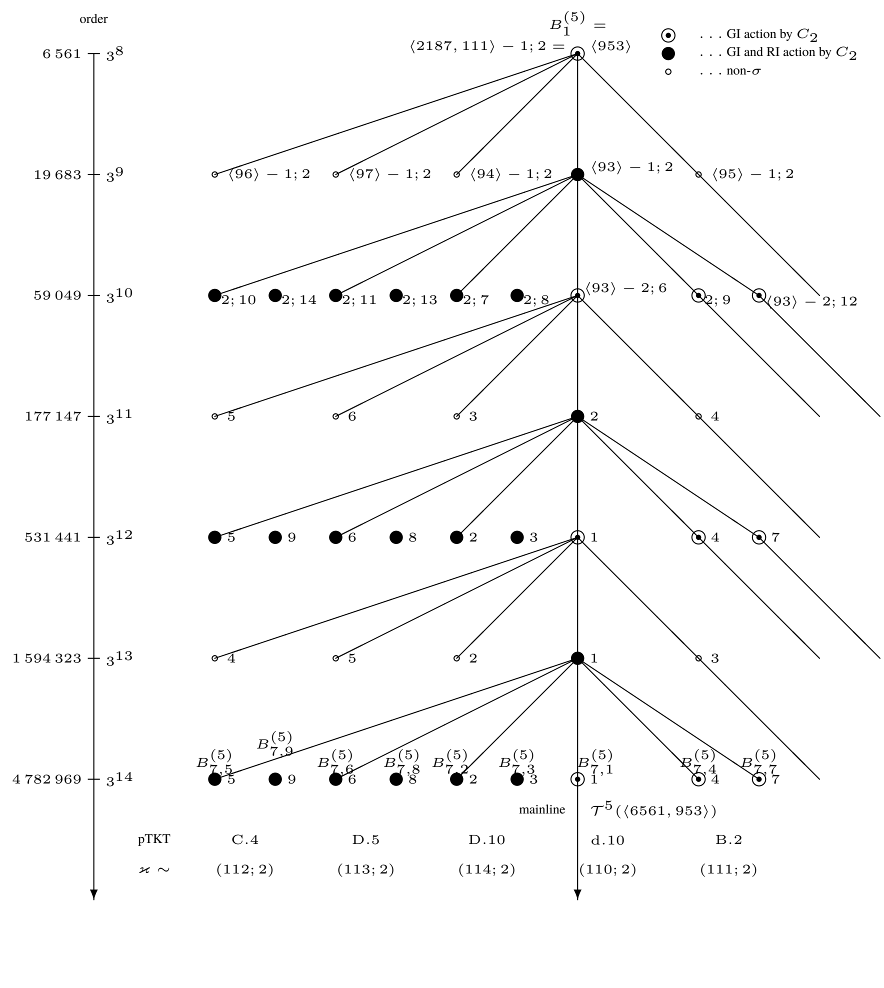
The polarization of all CF-descendants of $\langle 243,17\rangle = \langle 81,3\rangle - \#1;6$ in Figure 2 was in the *third* component. In Figure 5, we see that, astonishingly, the polarization of all BCF-descendants of $\langle 729,13\rangle = \langle 81,3\rangle - \#2;10$ is in the *first* component, although both roots are constructed as immediate descendants of the same parent. However, this discrepancy does not really matter, since the coclass trees $\mathcal{T}^2(\langle 243,17\rangle)$ (CF) and $\mathcal{T}^3(\langle 729,13\rangle)$ (BCF) are actually completely independent, in contrast to the following coclass trees in Figure 6 and 7. Branch 2 is realized by $\mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ of imaginary quadratic fields K — see [9, Tbl. 4, p. 161, Fig. 5, p. 162], [9, Exm. 1, p. 167].

Fig. 6 Depth-pruned metabelian coclass-4 BCF-tree of type d.10 with AQI (31)



In Figure 6, the root $\langle 2187, 112\rangle = \langle 729, 7\rangle - \#1; 2$ is p-terminal, and all its immediate descendants on the *first* branch are step size-2 p-descendants of $\langle 729, 7\rangle$. This is the prototype of an application of Formula (41), $B_1^{(4)} = C_1^{(3)} - \#1; 2$, and (43), $B_{2,t-5}^{(4)} = C_1^{(3)} - \#2; t$, both for $e = e_0 = 3$, the latter for $t \in \{6, 7, 8, 9, 10\}$. The *second* branch is regular with root $\langle 6561, 98\rangle = \langle 729, 7\rangle - \#2; 6$, but its relative identifiers are exceptional, since $\langle 6561, 98\rangle - \#1; i$ with $1 \le i \le 2$ are non-metabelian and thus remain hidden in the metabelian skeleton. Beginning with the *third* branch, all *odd* branches are regular with regular relative identifiers, according to Formula (30). Beginning with the *fourth* branch, all *even* branches are regular with regular relative identifiers, according to Formula (32). Less explicitly, Formula (41) can be expressed by Formula (21), and Formula (43) by Formula (22). The propagation in regular branches is covered by Formula (23). Branch 2 is realized by $\mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ of imaginary quadratic fields K [9, Exm. 2, p. 168].

Fig. 7 Depth-pruned metabelian coclass-5 BCF-tree of type $\mathrm{d}.10$ with AQI (41)



In Figure 7, the root and the initial branches 1 and 2 of the BCF-tree $\mathcal{T}^5$ are constituted by immediate descendants of CF-groups. The *root* $\langle 6561, 953\rangle = \langle 2187, 111\rangle - \#1; 2$ propagates endo-genetically from the root of the CF-tree $\mathcal{T}^4$ with (logarithmic) commutator quotient (41), according to Formula (21). The *first branch* comes from distinct mainline and offside vertices, $\langle 6561, i\rangle - \#1; 2$ with $93 \le i \le 97$, on the first branch of $\mathcal{T}^4$, according to the same Formula (21) with type change. The entire *second branch* uniformly propagates from the second mainline vertex of $\mathcal{T}^4$, according to Formula (22). Although branch 3 is regular, its relative identifiers are exceptional, since $\langle 6561, 93\rangle - \#2; 6 - \#1; 1$ is non-metabelian and thus does not show up in the metabelian skeleton. *Regular branches* (third, etc.) are constructed according to Formula (23). Branch 2 is realized by $\mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ of imaginary quadratic fields K [9, Exm. 3, p. 168].

9 Periodic Bifurcations and Periodic Chains

The statements in this section exhibit several new kinds of *periodicities* in p-descendant trees. The notations are based on both preceding sections, § 4 on CF-groups, and § 5 on BCF-groups.

Generally, it is convenient to view a coclass tree $\mathcal{T}^r$ as union of a finite *pre-period* $\mathcal{V}$ and an infinite disjoint union $\mathcal{T}^r = \mathcal{V}\dot\cup\bigl(\dot\bigcup_{k\ge 0}\mathcal{P}_k\bigr)$ of copies $\mathcal{P}_k = \dot\bigcup_{i=0}^{\ell-1}\mathcal{B}_{p+k\ell+i}$ of a collection of finitely many branches $\dot\bigcup_{i=0}^{\ell-1}\mathcal{B}_{p+i}$, the *period* with *length* $\ell \ge 1$ and *starting subscript* $p \ge 1$, such that the branches $\mathcal{B}_{p+k\ell+i} \simeq \mathcal{B}_{p+i}$, with $k \ge 0$, $0 \le i \le \ell - 1$, are isomorphic as finite graphs.

In the present work, all coclass trees are *depth*-1 *pruned metabelian skeletons* without pre-period, $\mathcal{V} = \emptyset$, minimal starting subscript $p = 1$, and period length $\ell = 2$, that is, we have $\mathcal{T}^r = \dot\bigcup_{k\ge 0}\mathcal{P}_k$ with $\mathcal{P}_k = \mathcal{B}_{1+2k}\dot\cup\mathcal{B}_{2+2k}$.

Definition 7 For each integer $i \ge 1$, the finite subtree $\mathcal{B}(C_i^{(e)}) = \mathcal{T}^e(C_i^{(e)}) \setminus \mathcal{T}^e(C_{i+1}^{(e)})$ of the depth-pruned CF-coclass tree $\mathcal{T}^e(C_1^{(e)})$ is called *i-th depth-*1 *pruned branch*, and the finite subtree $\mathcal{B}(B_i^{(e+1)}) = \mathcal{T}^{e+1}(B_i^{(e+1)}) \setminus \mathcal{T}^{e+1}(B_{i+1}^{(e+1)})$ of the depth-pruned BCF-coclass tree $\mathcal{T}^{e+1}(B_1^{(e+1)})$, restricted to the metabelian skeleton, is called *i-th depth-*1 *pruned metabelian branch*.

From now on, we omit the phrase “depth-1 pruned metabelian”. The precise constitution of the branches in Definition 7 by CF-vertices, respectively BCF-vertices, is given experimentally:

Proposition 5 ***(Odd Branches)*** *Let $e \ge 2$ be a logarithmic integer exponent.*
For each ***odd*** *integer $i \ge 1$, the i-th CF-branch $\mathcal{B}(C_i^{(e)}) = \{C_i^{(e)}\} \cup \{(C_{i+1,j}^{(e)})_{j=2}^{7}\}$ consists of the mainline vertex $C_i^{(e)}$ (branch root) and its immediate step size-*1 *offside* ***descendants***

$$
\begin{aligned}
&C^{(e)}_{i+1,2} \textit{ of type } \mathrm{b}.16,\ \varkappa \sim (004;0),\\
&C^{(e)}_{i+1,3} \textit{ of type } \mathrm{a}.1,\ \varkappa = (000;0),\ \textit{capable twig root},\\
&C^{(e)}_{i+1,4} \textit{ of type } \mathrm{a}.1,\ \varkappa = (000;0),\ \textit{bicyclic center } \zeta \simeq (3^{e-1},3),\\
&C^{(e)}_{i+1,5} \textit{ of type } \mathrm{a}.1,\ \varkappa = (000;0),\ \textit{bicyclic center } \zeta \simeq (3^{e-1},3),\\
&C^{(e)}_{i+1,6} \textit{ of type } \mathrm{b}.3,\ \varkappa \sim (001;0),\ \textit{cyclic center } \zeta \simeq (3^{e}),\\
&C^{(e)}_{i+1,7} \textit{ of type } \mathrm{a}.1,\ \varkappa = (000;0),\ \textit{cyclic center } \zeta \simeq (3^{e}).
\end{aligned}
\tag{29}
$$

For each ***odd*** *integer* $i \ge 1$*, the* i*-th BCF-branch* $\mathcal{B}(B^{(e+1)}_i) = \{B^{(e+1)}_i\} \cup \{(B^{(e+1)}_{i+1,j})^5_{j=2}\}$ *consists of the mainline vertex* $B^{(e+1)}_i$ *(branch root) and its immediate step size-*1 *offside* ***descendants***

$$
\begin{aligned}
&B^{(e+1)}_{i+1,2} \textit{ of type } \mathrm{D}.10,\ \varkappa \sim (114;2),\\
&B^{(e+1)}_{i+1,3} \textit{ of type } \mathrm{B}.2,\ \varkappa \sim (111;2),\ \textit{capable twig root},\\
&B^{(e+1)}_{i+1,4} \textit{ of type } \mathrm{C}.4,\ \varkappa \sim (112;2),\\
&B^{(e+1)}_{i+1,5} \textit{ of type } \mathrm{D}.5,\ \varkappa \sim (113;2).
\end{aligned}
\tag{30}
$$

Proof See the proofs of Theorems 4 and 5. □

Remark 4 1. In order to be able to include the mainline vertex, we always assume $C^{(e)}_{i+1,1} = C^{(e)}_{i+1}$ of type $\mathrm{a}.1$ for CF-groups and $B^{(e+1)}_{i+1,1} = B^{(e+1)}_{i+1}$ of type $\mathrm{d}.10$ for BCF-groups (Propositions 5–6, where the ordering of the offside vertices usually coincides with the ordering in Figures 3–7).

2. The branches for even subscripts $i \ge 2$ have bigger cardinality (9 for CF, 9 for BCF in Proposition 6) than those for odd subscripts $i \ge 1$ (7 for CF, 5 for BCF in Proposition 5).

3. It must be emphasized very clearly that the (abstract) **descendants** in Propositions 5 – 6 are **not** p-descendants in the region **behind and on the shockwave**.

Proposition 6 ***(Even Branches)*** *Let* $e \ge 2$ *be a logarithmic integer exponent.*
For each ***even*** *integer* $i \ge 2$*, the* i*-th CF-branch* $\mathcal{B}(C^{(e)}_i) = \{C^{(e)}_i\} \cup \{(C^{(e)}_{i+1,j})^9_{j=2}\}$ *consists of the mainline vertex* $C^{(e)}_i$ *(branch root) and its immediate step size-*1 *offside* ***descendants***

$$\begin{aligned}
&C^{(e)}_{i+1,2} \textit{ of type } \mathrm{b}.16,\ \varkappa\sim(004;0),\\
&C^{(e)}_{i+1,3} \textit{ of type } \mathrm{a}.1,\ \varkappa=(000;0),\ \textit{capable twig root},\\
&C^{(e)}_{i+1,4} \textit{ of type } \mathrm{a}.1,\ \varkappa=(000;0),\ \textit{bicyclic center } \zeta\simeq(3^{e-1},1),\\
&C^{(e)}_{i+1,5} \textit{ of type } \mathrm{a}.1,\ \varkappa=(000;0),\ \textit{bicyclic center } \zeta\simeq(3^{e-1},1),\\
&C^{(e)}_{i+1,6} \textit{ of type } \mathrm{b}.3,\ \varkappa\sim(001;0),\ \textit{cyclic center } \zeta\simeq(3^{e}),\\
&C^{(e)}_{i+1,7} \textit{ of type } \mathrm{a}.1,\ \varkappa=(000;0),\ \textit{cyclic center } \zeta\simeq(3^{e}),\\
&C^{(e)}_{i+1,8} \textit{ of type } \mathrm{b}.3,\ \varkappa\sim(001;0),\ \textit{cyclic center } \zeta\simeq(3^{e}),\\
&C^{(e)}_{i+1,9} \textit{ of type } \mathrm{a}.1,\ \varkappa=(000;0),\ \textit{cyclic center } \zeta\simeq(3^{e}).
\end{aligned} \tag{31}$$

For each ***even*** *integer $i \ge 2$, the i-th BCF-branch $\mathcal{B}(B^{(e+1)}_i) = \{B^{(e+1)}_i\} \cup \{(B^{(e+1)}_{i+1,j})_{j=2}^{9}\}$ consists of the mainline vertex $B^{(e+1)}_i$ (branch root) and its immediate step size-1 offside* ***descendants***

$$\begin{aligned}
&B^{(e+1)}_{i+1,2} \textit{ of type } \mathrm{D}.10,\ \varkappa\sim(114;2),\\
&B^{(e+1)}_{i+1,3} \textit{ of type } \mathrm{D}.10,\ \varkappa\sim(114;2),\\
&B^{(e+1)}_{i+1,4} \textit{ of type } \mathrm{B}.2,\ \varkappa\sim(111;2),\ \textit{capable twig root},\\
&B^{(e+1)}_{i+1,5} \textit{ of type } \mathrm{C}.4,\ \varkappa\sim(112;2),\\
&B^{(e+1)}_{i+1,6} \textit{ of type } \mathrm{D}.5,\ \varkappa\sim(113;2),\\
&B^{(e+1)}_{i+1,7} \textit{ of type } \mathrm{B}.2,\ \varkappa\sim(111;2),\ \textit{capable twig root},\\
&B^{(e+1)}_{i+1,8} \textit{ of type } \mathrm{D}.5,\ \varkappa\sim(113;2),\\
&B^{(e+1)}_{i+1,9} \textit{ of type } \mathrm{C}.4,\ \varkappa\sim(112;2).
\end{aligned} \tag{32}$$

Proof See the proofs of Theorems 4 and 5. □

We are now in the position to clarify in depth the structure of *periodic bifurcations*, that is, periodic chains with constant step size $s = 2$. This phenomenon concerns only the links between CF-coclass trees.

Proposition 7 ***(Periodic Step Size-2 Trunk)*** *For each commutator quotient $C_{3^e} \times C_3$ with $e \ge 3$, the mainline of the unique CF-coclass tree $\mathcal{T}^e(C^{(e)}_1)$, which starts at the root $C^{(e)}_1$ with $\mathrm{cl} = 3$, $\mathrm{cc} = e$, $\mathrm{lo} = 3+e$, contains a unique vertex $C^{(e)}_{e-2}$ with bifurcation due to nuclear rank $n = 2$, and with $\mathrm{cl} = e$, $\mathrm{cc} = e$, $\mathrm{lo} = 2e$. The complete periodic size-2 trunk is given by $(C^{(e)}_{e-2})_{e\ge3}$.*

Proof By definition of the coclass tree $\mathcal{T}^e(C_1^{(e)})$, all of its vertices share the common coclass $\mathrm{cc} = e$. Each tree arises from a root $C_1^{(e)}$ with class $\mathrm{cl} = 3 = 1 + 2$, whence generally each mainline vertex $C_i^{(e)}$ is of class $\mathrm{cl} = i + 2$, for $i \geq 1$. In particular, the distinguished vertex $C_{e-2}^{(e)}$ has class $\mathrm{cc} = (e-2) + 2 = e$. Evidence of its elevated nuclear rank $n = 2$ will be provided by the following Theorem 4. The logarithmic order is always the sum $\mathrm{lo} = \mathrm{cl} + \mathrm{cc}$ of class and coclass. □

Each of the step sizes, $s = 1$ and $s = 2$, of the bifurcations generates both, exo-genetic and endo-genetic p-descendants. Whereas Propositions 5–6 are *not constructive*, the following Theorems 4 and 5 can be viewed as *deterministic laws for the construction* of vertices with the aid of the p-group generation algorithm [12, 13, 1, 2]. Both theorems are experimental.

Theorem 4 ***(Structure of Bifurcations)*** *For each integer $e \geq 3$, the distinguished CF-mainline vertex $C_{e-2}^{(e)}$, which possesses bifurcation, gives rise to*

- 1 *exo-, and* 8*, respectively* 10*, endo-genetic propagations with step size* $s = 1$*,*

$$\begin{aligned}
C_{e-2}^{(e+1)} &= C_{e-2}^{(e)} - \#1; 1 \textit{ mainline of type } \mathrm{a.1} \textbf{\textit{ (exo-genetic)}},\\
B_{e-2}^{(e+1)} &= C_{e-2}^{(e)} - \#1; 2 \textit{ mainline of type } \mathrm{d.10},\\
C_{e-1}^{(e)} &= C_{e-2}^{(e)} - \#1; 3 \textit{ mainline of type } \mathrm{a.1},\\
C_{e-1,i-2}^{(e)} &= C_{e-2}^{(e)} - \#1; i \textit{ offside with } 4 \leq i \leq 9 \ (e \textit{ odd}),\\
&\qquad \textit{respectively } 4 \leq i \leq 11 \ (e \textit{ even});
\end{aligned} \tag{33}$$

- 5 *exo-, and* 5*, respectively* 9*, endo-genetic propagations with step size* $s = 2$*,*

$$\begin{aligned}
C_{e-1}^{(e+1)} &= C_{e-2}^{(e)} - \#2; 1 \textit{ distinguished mainline } \mathrm{a.1} \textbf{\textit{ (exo-genetic)}},\\
C_{e-1,i}^{(e+1)} &= C_{e-2}^{(e)} - \#2; i \textit{ offside with } 2 \leq i \leq 5 \textbf{\textit{ (exo-genetic)}},\\
B_{e-1}^{(e+1)} &= C_{e-2}^{(e)} - \#2; 6 \textit{ mainline of type } \mathrm{d.10},\\
B_{e-1,i-5}^{(e+1)} &= C_{e-2}^{(e)} - \#2; i \textit{ offside with } 7 \leq i \leq 10 \ (e \textit{ odd}),\\
&\qquad \textit{respectively } 7 \leq i \leq 14 \ (e \textit{ even}).
\end{aligned} \tag{34}$$

In particular, by endo-genetic propagations, $C_{e-2}^{(e)}$ generates the complete branch $\mathcal{B}(C_{e-2}^{(e)})$ of CF-groups (with $s = 1$), and the complete branch $\mathcal{B}(B_{e-2}^{(e+1)})$ of BCF-groups (with $s = 2$), the branch roots $C_{e-2}^{(e)}$ and $B_{e-2}^{(e+1)}$ inclusively (i.e., all vertices with depth $0 \leq \mathrm{dp} \leq 1$).

In the following Theorem 5, we abstain from vertices $C^{(e)}_{e-2} - \#2;3$ with brushwood type $\mathrm{B}.2$ of high complexity, and we restrict our attention to vertices with types $\mathrm{d}.10$, $\mathrm{D}.10$, $\mathrm{C}.4$, and $\mathrm{D}.5$. Again, each member of the chains generates both, exo-genetic and endo-genetic p-descendants.

Theorem 5 ***(Periodic Step Size-*1 *Chains)*** *For each integer $e \ge 3$, the vertex*

- $C^{(e+1)}_{e-2} = C^{(e)}_{e-2} - \#1;1$ *mainline of type* $\mathrm{a}.1$ *gives rise to*

$$\begin{aligned} C^{(e+2)}_{e-2} &= C^{(e+1)}_{e-2} - \#1;1 \textit{ mainline of type } \mathrm{a}.1 \textbf{\textit{ (exo-genetic recursion)}}, \\ B^{(e+2)}_{e-1} &= C^{(e+1)}_{e-2} - \#1;2 \textit{ mainline of type } \mathrm{d}.10, \\ C^{(e+1)}_{e-1,i+3} &= C^{(e+1)}_{e-2} - \#1;i \textit{ offside of types } \mathrm{b}.3, \mathrm{a}.1, \textit{ with } 3 \le i \le 4, \\ C^{(e+1)}_{e-1,i+3} &= C^{(e+1)}_{e-2} - \#1;i \textit{ offside of types } \mathrm{b}.3, \mathrm{a}.1, \textit{ with } 5 \le i \le 6, \\ &\qquad \textit{only for even } e. \end{aligned} \tag{35}$$

- $C^{(e+1)}_{e-1,2} = C^{(e)}_{e-2} - \#2;2$ *offside of type* $\mathrm{b}.16$ *gives rise to*

$$\begin{aligned} C^{(e+2)}_{e-1,2} &= C^{(e+1)}_{e-1,2} - \#1;1 \textit{ offside of type } \mathrm{b}.16 \textbf{\textit{ (exo-genetic recursion)}}, \\ B^{(e+2)}_{e-1,2} &= C^{(e+1)}_{e-1,2} - \#1;2 \textit{ offside of type } \mathrm{D}.10, \\ B^{(e+2)}_{e-1,3} &= C^{(e+1)}_{e-1,2} - \#1;3 \textit{ offside of type } \mathrm{D}.10, \textit{ only for even } e. \end{aligned} \tag{36}$$

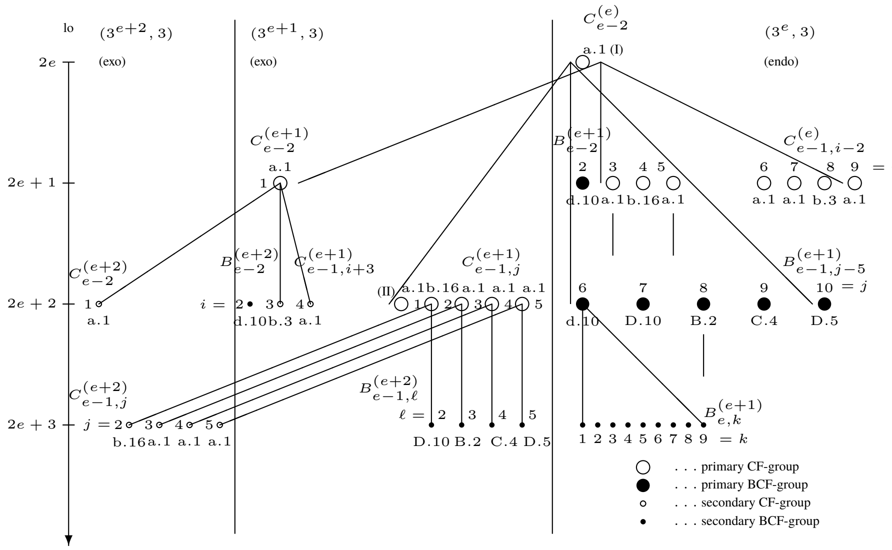
- $C^{(e+1)}_{e-1,4} = C^{(e)}_{e-2} - \#2;4$ *offside of type* $\mathrm{a}.1$ *gives rise to*

$$\begin{aligned} C^{(e+2)}_{e-1,4} &= C^{(e+1)}_{e-1,4} - \#1;1 \textit{ offside of type } \mathrm{a}.1 \textbf{\textit{ (exo-genetic recursion)}}, \\ B^{(e+2)}_{e-1,i} &= C^{(e+1)}_{e-1,4} - \#1;2 \textit{ offside of type } \mathrm{C}.4, \textit{ with } i = \begin{cases} 4 & \textit{e odd,} \\ 5 & \textit{e even,} \end{cases} \\ B^{(e+2)}_{e-1,9} &= C^{(e+1)}_{e-1,4} - \#1;3 \textit{ offside of type } \mathrm{C}.4, \textit{ only for even } e. \end{aligned} \tag{37}$$

- $C^{(e+1)}_{e-1,5} = C^{(e)}_{e-2} - \#2;5$ *offside of type* $\mathrm{a}.1$ *gives rise to*

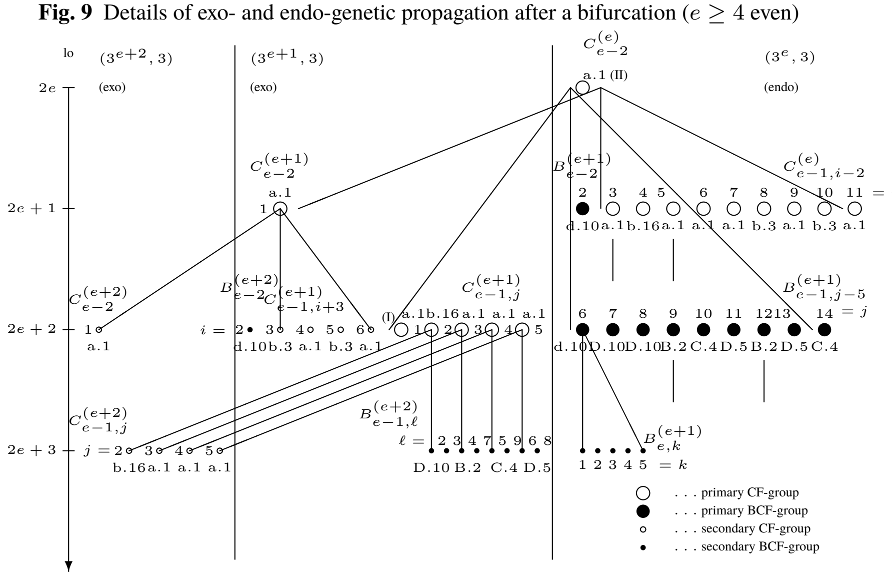
$$\begin{aligned} C^{(e+2)}_{e-1,5} &= C^{(e+1)}_{e-1,5} - \#1;1 \textit{ offside of type } \mathrm{a}.1 \textbf{\textit{ (exo-genetic recursion)}}, \\ B^{(e+2)}_{e-1,i} &= C^{(e+1)}_{e-1,5} - \#1;2 \textit{ offside of type } \mathrm{D}.5, \textit{ with } i = \begin{cases} 5 & \textit{e odd,} \\ 6 & \textit{e even,} \end{cases} \\ B^{(e+2)}_{e-1,8} &= C^{(e+1)}_{e-1,5} - \#1;3 \textit{ offside of type } \mathrm{D}.5, \textit{ only for even } e. \end{aligned} \tag{38}$$

Fig. 8 Details of exo- and endo-genetic propagation after a bifurcation ($e \geq 3$ odd)



Figures 8 and 9 provide a graphical illumination of the statements in Theorem 4 and 5. The periodicity of length two is indicated by roman numerals (I) and (II) at identification points.

Fig. 9 Details of exo- and endo-genetic propagation after a bifurcation ($e \geq 4$ even)



Definition 8 Let $n(P)$ be the nuclear rank of the finite p-group P. For $1 \le s \le n(P)$, we denote by $N_s(P)$ the number of all immediate step size-s p-descendants of P, and by $C_s(P) \le N_s(P)$ the number of those which are capable. The collection $(N_s(P)/C_s(P))_{1\le s\le n(P)}$ of pairs consisting of the numbers of all, respectively only the capable, p-descendants of P, for each possible step size s, is called the *propagation schema* of P.

Remember that an immediate step size-s p-descendant D of a p-parent group P is characterized by a *relative identifier* $D = P - \#s; j$ with $1 \le s \le n(P)$ and $1 \le j \le N_s(P)$ [12, 18].

Proof (Theorem 4) The distinguished mainline vertices $C^{(e)}_{e-2}$ of CF-trees $\mathcal{T}^e(C^{(e)}_1)$ form an infinite path of p-descendants, $(C^{(e)}_{e-2})_{e\ge 3}$, with alternating propagation schema $(9/4; 10/10)$ for odd exponents $e \in \{5, 7, 9, \ldots\}$, and $(11/4; 14/14)$ for even exponents $e \in \{2, 4, 6, \ldots\}$.
The immediate step size-1 p-descendants are:
$C^{(e)}_{e-2} - \#1; 1 = C^{(e+1)}_{e-2}$ CF-mainline vertex on the next CF-tree $\mathcal{T}^{e+1}(C^{(e+1)}_1)$,
$C^{(e)}_{e-2} - \#1; 2 = B^{(e+1)}_{e-2}$ BCF-mainline vertex on the associated BCF-tree $\mathcal{T}^{e+1}(B^{(e+1)}_1)$,
$C^{(e)}_{e-2} - \#1; 3 = C^{(e)}_{e-1}$ next CF-mainline vertex on the same CF-tree $\mathcal{T}^e(C^{(e)}_1)$,
$C^{(e)}_{e-2} - \#1; 4 = C^{(e)}_{e-1,2}$ leaf with $\varkappa \sim (004; 0)$, b.16,
$C^{(e)}_{e-2} - \#1; 5 = C^{(e)}_{e-1,3}$ twig root with $\varkappa \sim (000; 0)$, a.1,
$C^{(e)}_{e-2} - \#1; 6 = C^{(e)}_{e-1,4}$ leaf with $\varkappa \sim (000; 0)$, a.1, and bicyclic center $\zeta \simeq (3^{e-1}, 3)$,
$C^{(e)}_{e-2} - \#1; 7 = C^{(e)}_{e-1,5}$ another leaf with $\varkappa \sim (000; 0)$, a.1, and bicyclic center $\zeta \simeq (3^{e-1}, 3)$,
$C^{(e)}_{e-2} - \#1; 8 = C^{(e)}_{e-1,6}$ leaf with $\varkappa \sim (001; 0)$, b.3, and cyclic center $\zeta \simeq (3^e)$,
$C^{(e)}_{e-2} - \#1; 9 = C^{(e)}_{e-1,7}$ leaf with $\varkappa \sim (000; 0)$, a.1, and cyclic center $\zeta \simeq (3^e)$,
$C^{(e)}_{e-2} - \#1; 10 = C^{(e)}_{e-1,8}$ another leaf with $\varkappa \sim (001; 0)$, b.3, and $\zeta \simeq (3^e)$, only for e even,
$C^{(e)}_{e-2} - \#1; 11 = C^{(e)}_{e-1,9}$ another leaf with $\varkappa \sim (000; 0)$, a.1, and $\zeta \simeq (3^e)$, only for e even.
The immediate step size-2 p-descendants are:
$C^{(e)}_{e-2} - \#2; 1 = C^{(e+1)}_{e-1}$ next distinguished CF-mainline vertex on the next CF-tree $\mathcal{T}^{e+1}(C^{(e+1)}_1)$,
$C^{(e)}_{e-2} - \#2; 2 = C^{(e+1)}_{e-1,2}$ recursive root with $\varkappa \sim (004; 0)$, b.16,
$C^{(e)}_{e-2} - \#2; 3 = C^{(e+1)}_{e-1,3}$ recursive root with $\varkappa \sim (000; 0)$, a.1,
$C^{(e)}_{e-2} - \#2; 4 = C^{(e+1)}_{e-1,4}$ recursive root with $\varkappa \sim (000; 0)$, a.1,
$C^{(e)}_{e-2} - \#2; 5 = C^{(e+1)}_{e-1,5}$ recursive root with $\varkappa \sim (000; 0)$, a.1,
$C^{(e)}_{e-2} - \#2; 6 = B^{(e+1)}_{e-1}$ next BCF-mainline vertex on the associated BCF-tree $\mathcal{T}^{e+1}(B^{(e+1)}_1)$,
$C^{(e)}_{e-2} - \#2; 7 = B^{(e+1)}_{e-1,2}$ leaf with $\varkappa \sim (114; 2)$, D.10,

$C^{(e)}_{e-2}-\#2;8 = B^{(e+1)}_{e-1,3}$ twig root $\varkappa \sim (111;2)$, B.2 (e odd), other leaf $\varkappa \sim (114;2)$, D.10 (e even),
$C^{(e)}_{e-2} - \#2;9 = B^{(e+1)}_{e-1,4}$ leaf $\varkappa \sim (112;2)$, C.4 (e odd), twig root $\varkappa \sim (111;2)$, B.2 (e even),
$C^{(e)}_{e-2} - \#2;10 = B^{(e+1)}_{e-1,5}$ leaf $\varkappa \sim (113;2)$, D.5 (e odd), $\varkappa \sim (112;2)$, C.4 (e even),
$C^{(e)}_{e-2} - \#2;11 = B^{(e+1)}_{e-1,6}$ leaf with $\varkappa \sim (113;2)$, D.5, only for e even,
$C^{(e)}_{e-2} - \#2;12 = B^{(e+1)}_{e-1,7}$ another twig root with $\varkappa \sim (111;2)$, B.2, only for e even,
$C^{(e)}_{e-2} - \#2;13 = B^{(e+1)}_{e-1,8}$ another leaf with $\varkappa \sim (113;2)$, D.5, only for e even,
$C^{(e)}_{e-2} - \#2;14 = B^{(e+1)}_{e-1,9}$ another leaf with $\varkappa \sim (112;2)$, C.4, only for e even. □

Proof (Theorem 5) For the sake of brevity, we restricted the statements in the Theorem to the roots. Now we prove the results in full generality.
• The mainline vertex $C^{(e+1)}_{e-2} = C^{(e)}_{e-2} - \#1;1$ of the CF-tree $\mathcal{T}^{e+1}(C^{(e+1)}_1)$ gives rise to an infinite path of p-descendants, $(C^{(e+1+i)}_{e-2})_{i\ge 0} = (C^{(e+1)}_{e-2}[-\#1;1]^i)_{i\ge 0}$, with propagation schema $(4/2)$ for odd exponents $e \in \{5,7,9,\ldots\}$, and $(6/2)$ for even exponents $e \in \{2,4,6,\ldots\}$.
The immediate step size-1 p-descendants are:
$C^{(e+1+i)}_{e-2}-\#1;1 = C^{(e+2+i)}_{e-2}$ mainline of type $\mathrm{a}.1$ on the next CF-tree $\mathcal{T}^{e+2+i}(C^{(e+2+i)}_1)$,
$C^{(e+1+i)}_{e-2} - \#1;2 = B^{(e+2+i)}_{e-1}$ mainline of type $\mathrm{d}.10$ on the associated BCF-tree $\mathcal{T}^{e+2+i}(B^{(e+2+i)}_1)$,
$C^{(e+1+i)}_{e-2} - \#1;j = C^{(e+1+i)}_{e-1,j+3}$ offside of types $\mathrm{b}.3, \mathrm{a}.1$, with $3 \le j \le 4$,
$C^{(e+1+i)}_{e-2} - \#1;j = C^{(e+1+i)}_{e-1,j+3}$ offside of types $\mathrm{b}.3, \mathrm{a}.1$, with $5 \le j \le 6$, only for even e.
• The offside vertices $C^{(e+1)}_{e-1,t} = C^{(e)}_{e-2} - \#2;t$ of the CF-tree $\mathcal{T}^{e+1}(C^{(e+1)}_1)$ give rise to three infinite paths, $t \in \{2,4,5\}$, of p-descendants, $(C^{(e+1+i)}_{e-1,t})_{i\ge 0} = (C^{(e+1)}_{e-1,t}[-\#1;1]^i)_{i\ge 0}$, with propagation schema $(2/2)$ for odd exponents $e \in \{3,5,7,\ldots\}$, and $(3/3)$ for even exponents $e \in \{2,4,6,\ldots\}$.
The immediate step size-1 p-descendants for $t = 2$ are:
$C^{(e+1+i)}_{e-1,2}-\#1;1 = C^{(e+2+i)}_{e-1,2}$ offside of type $\mathrm{b}.16$ on the next CF-tree $\mathcal{T}^{e+2+i}(C^{(e+2+i)}_1)$,
$C^{(e+1+i)}_{e-1,2} - \#1;2 = B^{(e+2+i)}_{e-1,2}$ offside of type $\mathrm{D}.10$ on the associated BCF-tree $\mathcal{T}^{e+2+i}(B^{(e+2+i)}_1)$,
$C^{(e+1+i)}_{e-1,2} - \#1;3 = B^{(e+2+i)}_{e-1,3}$ offside of type $\mathrm{D}.10$, only for even e.
The immediate step size-1 p-descendants for $t = 4$ are:
$C^{(e+1+i)}_{e-1,4}-\#1;1 = C^{(e+2+i)}_{e-1,4}$ offside of type $\mathrm{a}.1$ on the next CF-tree $\mathcal{T}^{e+2+i}(C^{(e+2+i)}_1)$,
$C^{(e+1+i)}_{e-1,4} - \#1;2 = B^{(e+2+i)}_{e-1,j}$ offside of type C.4 on BCF-tree $\mathcal{T}^{e+2+i}(B^{(e+2+i)}_1)$, $j = 4$ or 5,
$C^{(e+1+i)}_{e-1,4} - \#1;3 = B^{(e+2+i)}_{e-1,9}$ offside of type C.4, only for even e.
The immediate step size-1 p-descendants for $t = 5$ are:
$C^{(e+1+i)}_{e-1,5}-\#1;1 = C^{(e+2+i)}_{e-1,5}$ offside of type $\mathrm{a}.1$ on the next CF-tree $\mathcal{T}^{e+2+i}(C^{(e+2+i)}_1)$,

$C_{e-1,5}^{(e+1+i)} - \#1;2 = B_{e-1,j}^{(e+2+i)}$ offside of type D.5 on BCF-tree $\mathcal{T}^{e+2+i}(B_1^{(e+2+i)})$, $j = 5$ or 6,
$C_{e-1,5}^{(e+1+i)} - \#1;3 = B_{e-1,8}^{(e+2+i)}$ offside of type D.5, only for even e.

We note that chains evolving from vertices $C_{e-2}^{(e)} - \#2;3$ of type B.2 possess high complexity with propagation schema $(11/4)$, respectively $(11/5)$, for roots, and $(6/2)$, respectively $(6/3)$, else. □

Eventually, we state the main theorem as the coronation of the present work.

Theorem 6 ***(Exhaustion Theorem)*** *Due to an infinite chain of periodic bifurcations, the p-descendant tree $\mathcal{T}_p(R)$ of the metabelian root $R = \langle 729, 7\rangle$ with abelianization $R/R' \simeq C_{27} \times C_3$ includes as* ***subsets****, for every commutator quotient $C_{3^e} \times C_3$ with logarithmic exponent $e \geq 3$, all depth-pruned coclass trees $\mathcal{T}^e(C_1^{(e)})$ of CF-groups with rank distribution $\varrho \sim (2,2,3;3)$ and all metabelian skeletons of depth-pruned coclass trees $\mathcal{T}^{e+1}(B_1^{(e+1)})$ of BCF-groups with rank distribution $\varrho \sim (2,2,3;3)$. The former are of type* a.1*, $\varkappa = (0,0,0;0)$, the latter of type* d.10*, $\varkappa = (1,1,0;2)$. The depth-pruning process eliminates all vertices with depth* $\mathrm{dp} \geq 2$.

We point out that we cannot speak about subtrees, because the coclass trees are **completely disconnected** as subgraphs of p-descendants in the finite region behind the shock wave. The coclass trees are **not subtrees** of $\mathcal{T}_p(R)$ (the problem are the different edges, not the vertices).

Proof Let $e \geq 3$ be the logarithmic exponent of an assigned non-elementary bicyclic commutator quotient $C_{3^e} \times C_3$.

First, we show that all vertices of the CF-coclass tree $\mathcal{T}^e(C_1^{(e)})$ are p-descendants of the root $C_1^{(3)} \simeq \langle 729, 7\rangle$.

- Vertices ahead of the shockwave, with class $c > e$, are constructed as regular descendants with endo-genetic propagation by iteration of Formula (8) along the mainline, and a single application of Formula (12) for vertices off mainline.
- For $e \geq 4$, vertices on the shockwave, with class $c = e$, are constructed as singular p-descendants with exo-genetic propagation by a single application of Formula (7), if they are mainline, and Formula (11), if they are offside with bicyclic center. If they are offside with cyclic center, they are constructed as regular p-descendants with endo-genetic propagation by Formula (13).
- For $e \geq 4$, all roots of CF coclass trees, with class $c = 3$, are constructed as irregular p-descendants with exo-genetic propagation by iteration of Formula (6).
- In the case $e \geq 5$, vertices behind the shockwave, with class $3 < c < e$, are constructed as irregular p-descendants with exo-genetic propagation by iteration of Formula (6), if they are mainline, and Formula (10), if they are offside with

bicyclic center. If they are offside with cyclic center, they are constructed as regular p-descendants with endo-genetic propagation by Formula (13).

Second, we show that all vertices of the BCF-coclass tree $\mathcal{T}^{e+1}(B_1^{(e+1)})$ are also p-descendants of the same root $C_1^{(3)} \simeq \langle 729, 7\rangle$.

- Vertices ahead of the shockwave, with class $c > e+1$, are constructed as regular descendants with endo-genetic propagation by iteration of Formula (23) along the mainline, followed by a single application of the same Formula (23) for vertices off mainline.
- Vertices on the shockwave, with class $c = e+1$, are constructed as singular p-descendants with endo-genetic propagation by a single application of Formula (22).
- All roots of BCF coclass trees, with class $c = 3$, are constructed as irregular p-descendants with endo-genetic propagation by a single application of Formula (21).
- In the case $e \geq 4$, vertices behind the shockwave, with class $3 < c < e+1$, are constructed as irregular p-descendants with endo-genetic propagation by a single application of Formula (21).

By the preceding distiction of cases, all claimed metabelian depth-pruned vertices are exhausted. □

The **Exhaustion Theorem** can be viewed from another perspective: instead of recursion formulas, completely explicit instructions are given for the construction of vertices on coclass trees of CF-groups and BCF-groups. Assume e_0 is a starting exponent and $e \geq e_0$ is a variable exponent.

1. For any $e_0 \geq 3$ (odd or even),

CF-groups are constructed as vertices on mainlines of type $\mathrm{a}.1$,

$$C_{e_0-2}^{(e)} = C_{e_0-2}^{(e_0)}[-\#1;1]^{e-e_0},\ e \geq e_0, \tag{39}$$

and as offside vertices of types $\mathrm{b}.16$, $\mathrm{a}.1$ twig, and two $\mathrm{a}.1$ with bicyclic center,

$$C_{e_0-1,t}^{(e)} = C_{e_0-2}^{(e_0)} - \#2;t[-\#1;1]^{e-(e_0+1)},\ e \geq e_0+1,\ t \in \{\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5}\}; \tag{40}$$

BCF-groups are constructed as vertices on mainlines of type $\mathrm{d}.10$,

$$B_{e_0-2}^{(e+1)} = C_{e_0-2}^{(e_0)}[-\#1;1]^{e-e_0} - \#1;\mathbf{2},\ e \geq e_0. \tag{41}$$

2. For $e_0 \geq 3$ **odd**,

CF-groups are constructed as offside vertices of types $\mathrm{b}.3$ and $\mathrm{a}.1$ with cyclic center,

$$C_{e_0-1,j+3}^{(e)} = C_{e_0-2}^{(e_0)}[-\#1;1]^{e-e_0} - \#1;j,\ e \geq e_0+1,\ j \in \{\mathbf{3},\mathbf{4}\}; \tag{42}$$

BCF-groups are constructed as offside vertices of types $\mathrm{D}.10$, $\mathrm{B}.2$, $\mathrm{C}.4$, $\mathrm{D}.5$,

$$B^{(e+1)}_{e_0-1,t} = C^{(e_0)}_{e_0-2} - \#2; t[-\#1;1]^{e-(e_0+1)} - \#1; \mathbf{2},\ e \geq e_0+1,\ t \in \{\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5}\}. \tag{43}$$

3. For $e_0 \geq 4$ **even**,

CF-groups are constructed as offside vertices of types $\mathrm{b}.3$, $\mathrm{a}.1$, $\mathrm{b}.3$, and $\mathrm{a}.1$ with cyclic center,

$$C^{(e)}_{e_0-1,j+3} = C^{(e_0)}_{e_0-2}[-\#1;1]^{e-e_0} - \#1; j,\ e \geq e_0+1,\ j \in \{\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}\}; \tag{44}$$

BCF-groups are constructed as offside vertices of types $\mathrm{D}.10$, $\mathrm{B}.2$, $\mathrm{C}.4$, $\mathrm{D}.5$,

$$B^{(e+1)}_{e_0-1,k_{tj}} = C^{(e_0)}_{e_0-2} - \#2; t[-\#1;1]^{e-(e_0+1)} - \#1; j, \tag{45}$$

where $e \geq e_0+1$, $j \in \{\mathbf{2},\mathbf{3}\}$, $t \in \{\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5}\}$, and

$$k_{tj} = \begin{cases} j & \text{if } t=2, \\ 4 & \text{if } t=3,\ j=2, \\ 7 & \text{if } t=3,\ j=3, \\ 5 & \text{if } t=4,\ j=2, \\ 9 & \text{if } t=4,\ j=3, \\ 6 & \text{if } t=5,\ j=2, \\ 8 & \text{if } t=5,\ j=3. \end{cases} \tag{46}$$

10 Extension and Unification of Excited States

The results concerning periodic *non-metabelian* Schur σ-groups G with moderate rank distribution $\varrho(G)$ and types $\mathrm{D}.10$, $\mathrm{C}.4$, $\mathrm{D}.5$ in [7, 8, 9] can be restated, extended, and unified in the terminology and notation of the present work. Periodic chains of both step sizes $s \in \{1,2\}$ must be employed, bifurcations with step size $s=2$ for the selection of *excited states* $n \geq 0$, and chains with step size $s=1$ for growing commutator quotients $(e,1)$ with logarithmic exponents $e \geq 2$.

10.1 Ground State

Pairs of periodic Schur σ-groups for the *ground state*, $n=0$, were discovered in [9, § 9, Thm. 12, Eqn. (9.1)–(9.3)]. For each of the types $\mathrm{D}.10$, $\mathrm{C}.4$, $\mathrm{D}.5$, determined by the fixed parameter $t \in \{2,4,5\}$, they were given by the sequence of doublets

$G = G_{e,i} = \langle 3^8, 93\rangle - \#2; t[-\#1; 1]^{e-5} - \#1; i - \#1; 1$ with running parameter $e \geq 5$ and selector $i \in \{2, 3\}$.

The constitution by an infinite main trunk and finite twigs was illuminated more closely in [8, § 4, Thm. 3–6, Eqn. (11)–(22)]. For each $t \in \{2, 4, 5\}$, a periodic chain of CF-groups $T_e = \langle 3^8, 93\rangle - \#2; t[-\#1; 1]^{e-5}$ with $e \geq 5$ forms the *trunk* of type b.16 for $t = 2$, and of type a.1 for $t \in \{4, 5\}$. Each of these vertices T_e gives rise to a finite *double twig of depth two*, consisting of BCF-groups, the metabelianizations $M_{e,i} = T_e - \#1; i \simeq G_{e,i}/G''_{e,i}$ with $i \in \{2, 3\}$ in depth one, and the Schur σ-groups $G_{e,i} = M_{e,i} - \#1; 1$ in depth two. The type is D.10 for $t = 2$, C.4 for $t = 4$, and D.5 for $t = 5$.

In both previous papers [8, 9], a connection between the ground state $G_{e,i}$ and branches of BCF-coclass trees, $B^{(e+1)}_{3,k_{tj}}$, is missing. The completely explicit notation of the present work, $T_e = C^{(e)}_{3,t}$, $M_{e,i} = B^{(e+1)}_{3,k_{tj}}$ admits the following restatement of all facts concerning the *ground state*.

Theorem 7 *The metabelianizations of the* ***ground state*** *of Schur σ-groups with type* D.10 *for $t = 2$,* C.4 *for $t = 4$, and* D.5 *for $t = 5$ are given by*

$$B^{(e+1)}_{3,k_{tj}} = C^{(e)}_{3,t} - \#1; j \quad \text{with} \quad j \in \{2, 3\}, \quad C^{(e)}_{3,t} = \langle 3^8, 93\rangle - \#2; t[-\#1; 1]^{e-5}, \tag{47}$$

for each $e \geq 5$. The subscript k_{tj} is given by Formula (46)*. $C^{(e)}_{3,t}$ belongs to the second branch $\mathcal{B}(C^{(e)}_2)$ of the CF-coclass tree $\mathcal{T}^e(C^{(e)}_1)$, and $B^{(e+1)}_{3,k_{tj}}$ belongs to the second branch $\mathcal{B}(B^{(e+1)}_2)$ of the BCF-coclass tree $\mathcal{T}^{e+1}(B^{(e+1)}_1)$. The Schur σ-group $B^{(e+1)}_{3,k_{tj}} - \#1; 1$ has soluble length* $\mathrm{sl} = 3$.

Remark 5 So the new insight in comparison to [8, 9] is that the exo-genetic propagation behind the shockwave establishes a branchwise mapping $C^{(e)}_{3,t} \mapsto \left(B^{(e+1)}_{3,k_{t2}}, B^{(e+1)}_{3,k_{t3}}\right)$ from the CF-coclass tree $\mathcal{T}^e(C^{(e)}_1)$ to the BCF-coclass tree $\mathcal{T}^{e+1}(B^{(e+1)}_1)$, for each $e \geq 5$.

Proof With respect to Schur σ-groups as possible p-descendants, only distinguished CF-mainline vertices $C^{(e)}_{e-2}$ with *even* coclass $e \geq 4$ are relevant. For the *ground* state, we need the smallest even bifurcation $C^{(4)}_2$ with $e = 4$ and exo-genetic offside p-descendants $C^{(5)}_{3,t} = C^{(4)}_2 - \#2; t$ with types b.16 for $t = 2$, and a.1 for $t \in \{4, 5\}$, each of them root of a periodic chain with step size $s = 1$, namely $C^{(e)}_{3,t} = C^{(5)}_{3,t}[-\#1; 1]^{e-5}$ for $e \geq 5$, according to Formula (40). These CF-groups give rise to pairs of BCF-groups as endo-genetic p-descendants $B^{(6)}_{3,k_{tj}} = C^{(5)}_{3,t} - \#1; j$, and more generally, for $e \geq 5$, $B^{(e+1)}_{3,k_{tj}} = C^{(e)}_{3,t} - \#1; j$ with $j \in \{2, 3\}$, according to Formula (45). In the SmallGroups library [14], $C^{(4)}_2$ has the absolute identifier $\langle 3^8, 93\rangle$, which completes the proof. □

10.2 First Excited State

Pairs of periodic Schur σ-groups for the *first excited state*, $n = 1$, were discovered in [7, § 2, Thm. 2, Eqn. (2)–(4)]. For each of the types $\mathrm{D}.10$, $\mathrm{C}.4$, $\mathrm{D}.5$, determined by the fixed parameter $\ell \in \{2,4,5\}$, they were given by the sequence of doublets $G = G_{e,i} \simeq M_{e,i}[-\#1;1]^2$ with metabelianization $M_{e,i} = G_{e,i}/G''_{e,i} \simeq W_\ell[-\#1;1]^{e-7} - \#1;\ell$, where $e \geq 7$, $i \in \{2,3\}$, and $W_\ell = \langle 3^8, 93\rangle[-\#2;1]^2 - \#2;\ell$.

The constitution by an infinite main trunk and finite twigs was illuminated more closely in [8, § 5, Thm. 8–10, Eqn. (27)–(38)]. For each $t \in \{2,4,5\}$, a periodic chain of CF-groups $T_e = \langle 3^8, 93\rangle[-\#2;1]^2 - \#2;t[-\#1;1]^{e-7}$ with $e \geq 7$ forms the *trunk* of type $\mathrm{b}.16$ for $t = 2$, and of type $\mathrm{a}.1$ for $t \in \{4,5\}$. Each of these vertices T_e gives rise to a finite *double twig of depth three*, consisting of BCF-groups, the metabelianizations $M_{e,i} = T_e - \#1;i \simeq G_{e,i}/G''_{e,i}$ with $i \in \{2,3\}$ in depth one, and the Schur σ-groups $G_{e,i} = M_{e,i}[-\#1;1]^2$ in depth three. The type is $\mathrm{D}.10$ for $t = 2$, $\mathrm{C}.4$ for $t = 4$, and $\mathrm{D}.5$ for $t = 5$.

As before, in both previous papers [7, 8], a connection between the first excited state and branches of coclass trees is missing. Again, the completely explicit notation of the present work admits the following restatement of all facts concerning the *first excited state*.

Theorem 8 *The metabelianizations of the* ***first excited state*** *of Schur σ-groups with type* $\mathrm{D}.10$ *for $t = 2$,* $\mathrm{C}.4$ *for $t = 4$, and* $\mathrm{D}.5$ *for $t = 5$ are given by*

$$\begin{aligned} B^{(e+1)}_{5,k_{tj}} &= C^{(e)}_{5,t} - \#1;j \quad \textit{with} \quad j \in \{2,3\}, \\ C^{(e)}_{5,t} &= \langle 3^8, 93\rangle[-\#2;1]^2 - \#2;t[-\#1;1]^{e-7}, \end{aligned} \tag{48}$$

for each $e \geq 7$. The subscript k_{tj} is given by Formula (46). $C^{(e)}_{5,t}$ belongs to the fourth branch $\mathcal{B}(C^{(e)}_4)$ of the CF-coclass tree $\mathcal{T}^e(C^{(e)}_1)$, and $B^{(e+1)}_{5,k_{tj}}$ belongs to the fourth branch $\mathcal{B}(B^{(e+1)}_4)$ of the BCF-coclass tree $\mathcal{T}^{e+1}(B^{(e+1)}_1)$. The Schur σ-group $B^{(e+1)}_{5,k_{tj}}[-\#1;1]^2$ has soluble length $\mathrm{sl} = 3$.

Remark 6 Again, the new insight in comparison to [7, 8] is that the exo-genetic propagation behind the shockwave establishes a branchwise mapping $C^{(e)}_{5,t} \mapsto \bigl(B^{(e+1)}_{5,k_{t2}}, B^{(e+1)}_{5,k_{t3}}\bigr)$ from the CF-coclass tree $\mathcal{T}^e(C^{(e)}_1)$ to the BCF-coclass tree $\mathcal{T}^{e+1}(B^{(e+1)}_1)$, for each $e \geq 7$.

Proof For the *first excited* state, we need the next even bifurcation $C^{(6)}_4$ with $e = 6$ and exo-genetic offside p-descendants $C^{(7)}_{5,t} = C^{(6)}_4 - \#2;t$ with types $\mathrm{b}.16$ for $t = 2$, and $\mathrm{a}.1$ for $t \in \{4,5\}$, according to Formula (40). These CF-groups give rise to pairs of BCF-groups as endo-genetic p-descendants $B^{(8)}_{5,k_{tj}} = C^{(7)}_{5,t} - \#1;j$ with $j \in \{2,3\}$, according to Formula (45). In order to start within the SmallGroups database [14], we observe that $C^{(6)}_4 = C^{(4)}_2[-\#2;1]^2$, where $C^{(4)}_2 \simeq \langle 3^8, 93\rangle$. □

10.3 n-th Excited State

Now we can easily extend the previous results by generalization to the n-th excited state for $n \geq 2$. For the sake of completeness, we include $n = 0$ and $n = 1$.

Theorem 9 *The metabelianizations of the* **n*-th excited state*** *of Schur* σ*-groups with type* D.10 *for* $t = 2$, C.4 *for* $t = 4$, *and* D.5 *for* $t = 5$ *are given by*

$$\begin{aligned} B^{(e+1)}_{3+2n,k_{tj}} &= C^{(e)}_{3+2n,t} - \#1; j \text{ with } j \in \{2,3\}, \\ C^{(e)}_{3+2n,t} &= \langle 3^8, 93\rangle[-\#2;1]^{2n} - \#2; t[-\#1;1]^{e-(5+2n)}, \end{aligned} \tag{49}$$

for each $e \geq 5 + 2n$. *The subscript* k_{tj} *is given by Formula* (46). $C^{(e)}_{3+2n,t}$ *belongs to the* $2(n+1)$*-th branch* $\mathcal{B}(C^{(e)}_{2(n+1)})$ *of the CF-coclass tree* $\mathcal{T}^e(C^{(e)}_1)$, *and* $B^{(e+1)}_{3+2n,k_{tj}}$ *belongs to the* $2(n+1)$*-th branch* $\mathcal{B}(B^{(e+1)}_{2(n+1)})$ *of the BCF-coclass tree* $\mathcal{T}^{e+1}(B^{(e+1)}_1)$. *The Schur* σ*-group* $B^{(e+1)}_{3+2n,k_{tj}}[-\#1;1]^{n+1}$ *has soluble length* $\mathrm{sl} = 3$.

Proof By induction with respect to the excited state $n \geq 2$, using Theorem 7 for $n = 0$ and Theorem 8 for $n = 1$ as induction hypothesis. The second derived subgroup $G'' > 1$ of all Schur σ-groups G for arbitrary logarithmic exponent $e \geq 2$, arbitrary type C.4, D.5, D.10, i.e., $t \in \{2, 4, 5\}$, and arbitrary state $n \geq 0$ is non-trivial but contained in the center $\zeta(G)$ of G. Hence, G'' is abelian, the third derived subgroup $G''' = 1$ is trivial, and the soluble length is $\mathrm{sl}(G) = 3$. □

This crucial result admits an important number theoretic conclusion.

Corollary 1 *The automorphism group* $\mathrm{Gal}(\mathrm{F}^\infty_3(K)/K)$ *of the maximal unramified pro-3-extension of any imaginary quadratic number field* $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, *with non-elementary bicyclic 3-class group* $\mathrm{Cl}_3(K) \simeq (\mathbb{Z}/3^e\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$, $e \geq 2$, *and capitulation type* C.4, D.5 *or* D.10, *can be constructed by Equation* (49) *in Theorem 9, and the Hilbert 3-class field tower of* K *has precise length* $\ell(K) = 3$, *that is, three stages* $K = \mathrm{F}^0_3(K) < \mathrm{F}^1_3(K) < \mathrm{F}^2_3(K) < \mathrm{F}^3_3(K) = \mathrm{F}^\infty_3(K)$.

11 Parents of Class Two

In the proofs of Theorem 1 and 3, we had to exclude the investigation of parents $A = \pi(D) = D/\gamma_c(D)$ of the roots $D = C_1^{(e)}$ respectively $D = B_1^{(e+1)}$ of coclass trees for $e \geq 2$. In Lemma 1, we construct a periodic chain with step size $s = 1$, which consists precisely of these parents. Since the distinction between CF- and BCF-groups begins with class three, the parents $\pi(D)$ are neither CF- nor BCF-groups but simply *class two*-groups.

Lemma 1 ***(Unboundedly extensible*** 3***-groups of class*** 2***)***
For each logarithmic exponent $e \geq 2$*, the* ***unique*** *infinitely capable* 3*-group* Θ_e *of class* 2 *and type* $\mathrm{a}.1$, $\varkappa = (0,0,0;0)$*, with commutator quotient* $C_{3^e} \times C_3$ *is given as a member of a periodic chain with step size* $s = 1$*. It is parent (but* ***not*** p*-parent for* $e \geq 3$*) of both,* $C_1^{(e)}$ *and* $B_1^{(e+1)}$.

$$\Theta_e := \Theta_2[-\#1;1]^{e-2}, \quad \pi(C_1^{(e)}) = \Theta_e, \quad \pi(B_1^{(e+1)}) = \Theta_e, \qquad (50)$$

where $\Theta_2 \simeq \mathrm{SmallGroup}(81,3)$ *denotes the root of the chain.*

Remark 7 Aside from the root $\Theta_2 \simeq \langle 81,3\rangle$, the SmallGroups database [14] also contains $\Theta_3 \simeq \langle 243,12\rangle$, $\Theta_4 \simeq \langle 729,61\rangle$, $\Theta_5 \simeq \langle 2187,315\rangle$, and $\Theta_6 \simeq \langle 6561,2063\rangle$. As the elementary analogue, we can view the extra special group $\Theta_1 \simeq \langle 27,3\rangle$ with commutator quotient $C_3 \times C_3$.

Proof With our usual convention $s_2 = [y,x]$ for the main commutator of a finite two-generated 3-group $G = \langle x,y\rangle$, a parametrized pc-presentation of all members of the chain is given by

$$\Theta_e = \langle x,y \mid x^{3^{e-1}} = w,\ w^3 = 1,\ y^3 = 1,\ s_2^3 = 1\rangle. \qquad (51)$$

Whereas the nilpotency class of all members is constant $\mathrm{cl}(\Theta_e) = 2$, the p-class $c_p = \mathrm{cl}_p(\Theta_e) = e$ depends on the logarithmic exponent e. Since the last non-trivial lower exponent-p central is $P_{c_p-1}(\Theta_e) = \langle w\rangle$, it follows that $\pi_p(\Theta_e) = \Theta_e/P_{e-1}(\Theta_e) \simeq \Theta_{e-1}$ for $e \geq 3$. The actual implementation in Magma [18] shows that $\Theta_e = \Theta_{e-1} - \#1;1$ for $e \geq 3$, and thus by induction $\Theta_e = \Theta_2(-\#1;1)^{e-2}$ for $e \geq 2$.

Now we come to the justification of the parent relations. First observe that Formula (14) degenerates to

$$C_1^{(e)} = \langle x,y \mid\ x^{3^{e-1}} = w,\ w^3 = 1,\ y^3 = 1,\ s_2^3 = s_3^3 = 1,\ t_3 = s_3,\ t_4 = s_4 = 1\,\rangle. \qquad (52)$$

in the special case of the root with class $c = 3$, for each $e \geq 2$. We put $D = C_1^{(e)}$. For $e \geq 4$, we are in the irregular region behind the shock wave, and we have $c = 3$,

$c_p = e$, $\gamma_3(D) = \langle s_3 \rangle$ and $P_{e-1}(D) = \langle w \rangle$, whence $A = \pi(D) = D/\gamma_3(D) \simeq \Theta_e$, as claimed, and $A_p = \pi_p(D) = D/P_{e-1}(D) \simeq C_1^{(e-1)}$, as known from Formula (6). For $e = 3$, the behavior on the shock wave is singular, i.e. $c = c_p = 3$, but $\gamma_3(D) = \langle s_3 \rangle$ as opposed to $P_2(D) = \langle s_3, w \rangle$. Thus $A = \pi(D) = D/\gamma_3(D) \simeq \Theta_3$, but $A_p = \pi_p(D) = D/P_2(D) \simeq \Theta_2$, due to bifurcation. For $e = 2$, the situation is regular (ahead of the shock wave), i.e. $c = c_p = 3$, $\gamma_3(D) = P_2(D) = \langle s_3 \rangle$ and $A_p = \pi_p(D) = A = \pi(D) = D/\gamma_3(D) \simeq \Theta_2$.

On the other hand, note that Formula (24) degenerates to

$$B_1^{(e+1)} = \langle x, y \mid x^{3^e} = w,\ w^3 = 1,\ y^3 = 1,\ s_2^3 = s_3^3 = 1,\ t_3 = s_3 w,\ t_4 = s_4 = 1 \rangle. \tag{53}$$

in the special case of the root with class $c = 3$, for each $e \geq 2$. We put $D = B_1^{(e+1)}$. Then we have $c = 3$, $c_p = e + 1$, $\gamma_3(D) = \langle s_3, t_3 \rangle = \langle s_3, w \rangle$ and $P_e(D) = \langle w \rangle$, whence $A = \pi(D) = D/\gamma_3(D) \simeq \Theta_e$, as claimed, and $A_p = \pi_p(D) = D/P_e(D) \simeq C_1^{(e)}$, as known from Formula (21). □

12 Conclusion

Throughout this work, we have focussed on a purely group theoretic exposition of CF-groups and BCF-groups with moderate rank distribution $\varrho \sim (2, 2, 3; 3)$, and we have intentionally abandoned all number theoretic applications, except for a general motivation in the introductory section § 1, and several occasional references to imaginary quadratic fields K whose second 3-class groups $\mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ occur as vertices in the Figures with BCF-coclass trees of type d.10, $\varkappa \sim (1, 1, 0; 2)$.

Therefore we want to point out a summary of arithmetical impacts arising from Theorem 3 and Theorem 9. The former admits the construction of the *second* 3-*class group* $\mathrm{Gal}(\mathrm{F}_3^2(K)/K)$, a metabelian BCF-group, of *any algebraic number field* K with capitulation type C.4, D.5 or D.10, according to Algorithm 1. The latter shows a way how to determine the 3-*class field tower group* $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$, a three-stage non-metabelian p-descendant of a BCF-group and a Schur σ-group, of any *imaginary quadratic* number field $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, with one of these capitulation types C.4, D.5 or D.10.

Remark 8 At the beginning of the Introduction § 1, difficulties in the construction of the groups $\mathrm{G}_3^2(K) = \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ and $\mathrm{G}_3^\infty(K) = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ for imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with 3-class groups $\mathrm{Cl}_3(K) \simeq (3^e, 3)$, $e \geq 4$, were indicated as a motivation for this work. Ostensively, the change from well-behaved to unusual techniques for the construction can be illuminated with examples for $e = 3$ and $e = 4$. According to [9, Thm. 9, p. 167, Exm. 2, p. 168], the discriminants $d \in \{-41\,631, -110\,059, -382\,232\}$ give rise to punctured transfer kernel type (pTKT) $\mathrm{C}.4$, $\mathrm{D}.10$, $\mathrm{D}.5$, respectively. In terms of the bifurcation $B = \langle 3^8, 98\rangle$, the groups of $K = \mathbb{Q}(\sqrt{-41\,631})$ are $\mathrm{G}_3^2(K) = B-\#1;i$ with $i \in \{7, 11\}$ in Figure 6, and $\mathrm{G}_3^\infty(K) = B - \#2;j$ with $j \in \{5, 9\}$, a **sibling** of step size $s = 2$ of $\mathrm{G}_3^2(K)$, as known from $e = 1, 2$. However, according to [9, Thm. 10, Exm. 3, p. 168], the discriminants $d \in \{-469\,283, -584\,411, -617\,363\}$ are associated with pTKT $\mathrm{D}.10$, $\mathrm{D}.5$, $\mathrm{C}.4$, respectively. In terms of the group $A_1 = \langle 3^8, 93\rangle$, the groups of $K = \mathbb{Q}(\sqrt{-469\,283})$ are $\mathrm{G}_3^2(K) = A_1 - \#2;i$ with $i \in \{7, 8\}$ in Figure 7, and $\mathrm{G}_3^\infty(K) = \mathrm{G}_3^2(K) - \#1;1$, a **descendant** of $\mathrm{G}_3^2(K)$, as a completely new phenomenon for $e = 4$.

In both construction processes, the *scaffold* of CF-groups turned out to be an essential tool which provides the p-paths to the desired BCF-groups, although the CF-groups themselves are irrelevant for imaginary quadratic fields, since they can only be realized by real quadratic fields. Type $\mathrm{B}.2$ has been excluded, since its descendants reveal high complexity and no statements concerning the precise length $\ell(K) \geq 3$ of 3-class field towers over imaginary quadratic fields K are possible. Any realization of the types $\mathrm{C}.4$, $\mathrm{D}.5$, $\mathrm{D}.10$ by imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, independently of the state, however, has a 3-class field tower with exact length $\ell(K) = 3$, underpinned by all numerical examples of ground states in [9], and by the few known examples of first excited states of type $(9, 3)$, $d = -320\,968$ $\mathrm{C}.4$, $d = -354\,232$ $\mathrm{D}.10$, and $d = -776\,747$ $\mathrm{D}.5$.

13 Outlook

1. In view of future research, it should be emphasized that a similar set of theorems can be proved for the root $\dot{C}_1^{(3)} = \langle 729, 6\rangle$, the analogue of **Ascione's CF-group G** [4] for the commutator quotient $(27, 3)$, which gives rise to infinitely many CF-coclass trees $\mathcal{T}^e(\dot{C}_1^{(e)})$, $e \geq 3$, of the same type $\mathrm{a}.1$, $\varkappa = (0, 0, 0; 0)$, and to infinitely many **pairs of BCF-coclass trees** $\mathcal{T}^{e+1}(\dot{B}_1^{(e+1)})$ and $\mathcal{T}^{e+1}(\ddot{B}_1^{(e+1)})$, $e \geq 3$, of type $\mathrm{e}.14$, $\varkappa = (1, 3, 2; 0)$ (para), respectively $\varkappa = (1, 2, 3; 0)$ (ortho), all three with a distinct rank distribution $\varrho \sim (2, 2, 2; 3)$. As opposed to the trees in the present work, the *polarization* for all these trees coincides with the *puncture* at the fourth component.
2. Concerning the present work, since the mainline of a coclass tree, $\mathcal{T}^e(C_1^{(e)})$ respectively $\mathcal{T}^{e+1}(B_1^{(e+1)})$, gives rise to an infinite projective limit of the same coclass, $C_\infty^{(e)} = \varprojlim_{i\geq 1} C_i^{(e)}$ respectively $B_\infty^{(e+1)} = \varprojlim_{i\geq 1} B_i^{(e+1)}$, it would be interesting to investigate whether the particular limit $C_\infty^{(3)}$ "generates" all limit groups $C_\infty^{(e)}$ with $e \geq 4$ and $B_\infty^{(e+1)}$ with $e \geq 3$, in some sense.
3. Stimulated by a question of the referee, the author has verified by means of the p-group generation algorithm [13, 1, 2] that similar connections between CF-groups and BCF-groups also exist for $p = 5$ and $p = 7$. The details, however, shall be investigated in a future paper.

Acknowledgements The author expresses his gratitude to the anonymous referees for valuable suggestions concerning the layout and interesting stimulations of future research. He acknowledges that his research was supported by the Austrian Science Fund (FWF): projects J0497-PHY, P26008-N25, and by the Research Executive Agency of the European Union (EUREA): project Horizon Europe 2021–2027.

References

1. Newman, M.F.: Determination of groups of prime-power order. In: Group Theory, Canberra, 1975, Lecture Notes in Math., Vol. **573**, pp. 73–84. Springer, Berlin (1977)
2. O'Brien, E.A.: The p-group generation algorithm. J. Symbolic Comput. **9**, 677–698 (1990)
3. Mayer, D.C.: Periodic bifurcations in descendant trees of finite p-groups. Adv. Pure Math. **5**, 162–195 (2015) doi: 10.4236/apm.2015.54020
4. Ascione, J.A., Havas, G., Leedham-Green, C.R.: A computer aided classification of certain groups of prime power order. Bull. Austral. Math. Soc. **17**, 257–274 (1977)
5. Nebelung, B.: Klassifikation metabelscher 3-Gruppen mit Faktorkommutatorgruppe vom Typ (3,3) und Anwendung auf das Kapitulationsproblem. Inauguraldissertation, Universität zu Köln (1989)
6. Mayer, D.C.: BCF-groups with elevated rank distribution. arXiv:2110.03558 (2021)
7. Mayer, D.C.: First excited state with moderate rank distribution. arXiv:2110.06511 (2021)
8. Mayer, D.C.: Periodic Schur sigma-groups of non-elementary bicyclic type. arXiv:2110.13886 (2021)
9. Mayer, D.C.: Bicyclic commutator quotients with one non-elementary component. Math. Bohem. **148**, 149–180 (2023) doi: 10.21136/MB.2022.0127-21
10. Mayer, D.C.: Principalization algorithm via class group structure. Joint CSASC Conference, Danube University, Krems, Austria (2011)
11. Mayer, D.C.: Artin transfer patterns on descendant trees of finite p-groups. Adv. Pure Math. **6**, 66–104 (2016) doi: 10.4236/apm.2016.62008
12. Gamble, G., Nickel, W., O'Brien, E.A.: ANU p-Quotient — p-Quotient and p-Group Generation Algorithms. An accepted GAP package, available also in MAGMA (2006)
13. Holt, D.F., Eick, B., O'Brien, E.A.: Handbook of computational group theory. Discrete mathematics and its applications, Chapman and Hall/CRC Press, Boca Raton (2005)
14. Besche, H.U., Eick, B., O'Brien, E.A.: The SmallGroups Library — a Library of Groups of Small Order. An accepted and refereed GAP package, available also in MAGMA (2005)
15. Mayer, D.C.: Modeling rooted in-trees by finite p-groups. In: Sirmacek, B.(ed.), Graph Theory — Advanced Algorithms and Applications, pp. 85–113. InTech, Rijeka (2018) doi: 10.5772/intechopen.68703
16. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. The user language. J. Symbolic Comput. **24**, 235–265 (1997)
17. Bosma, W., Cannon, J.J., Fieker, C., Steel, A. (eds.): Handbook of Magma functions. Ed. 2.28, Sydney (2023)
18. The Magma Group: Magma Computational Algebra System. Version `2.28-2`, University of Sydney (2023)
19. James, R.: The groups of order p^6 ($p \geq 3$). Ph.D. Thesis, University of Sydney (1968)
20. James, R.: The groups of order p^6 (p an odd prime). Math. Comp. **34**, 613–637 (1980)