Der Wissenschaftsfonds.

Periodic Sequences of p-Class Tower Groups

Conference:	International Conference on Groups and Algebras (ICGA 2015)
Place:	Guang Dong Hotel, 328 Yixian Road, Hongkou
Venue:	Shanghai, China
Date:	Tuesday, July 21, 2015
Time:	09:45 – 10:00, a.m.
Author:	Daniel C. Mayer (Austria)

A presentation within the frame of the international scientific research project

Towers of p-Class Fields over Algebraic Number Fields

INTRODUCTION. SUCCINCT SURVEY

• The key for determining the Galois group

$$G := \mathrm{G}_p^\infty(K) = \mathrm{Gal}(\mathrm{F}_p^\infty(K)|K)$$

of the unramified Hilbert p-class field tower $\mathrm{F}_p^\infty(K)$, i.e. the maximal unramified pro-p extension, of an algebraic number field K is a thorough understanding of finite p-group theory.
G is briefly called p-**class tower group** of K.

• Our main goal is to draw the attention to a new kind of **periodic sequences** of p-class tower groups G with strictly **increasing coclass** $\mathrm{cc}(G)$.

• This presentation can be downloaded from
http://www.algebra.at/PresICGA2015.pdf

• It is a compact version of our paper

[T3] D.C. Mayer,
Periodic sequences of p-class tower groups,
J. Appl. Math. Phys. **3** (2015), 746–756,
DOI 10.4236/jamp.2015.37090.

p ... prime number,
G ... finite p-group with nuclear rank n and immediate descendant numbers $(N_1, \ldots, N_n)$.

Notation: For $1 \le s \le n$ and $1 \le i \le N_s$, the symbol

$$G - \#s; i$$

denotes the
ith *immediate descendant* of *step size* s of G.

Until August 2012, no p-class towers of finite length $\ell_p(K)$ bigger than 2 were known for odd $p \geq 3$.

Theorem 1. (Boston, Bush and Mayer, 2012) Let $p = 3$, $K = \mathbb{Q}(\sqrt{d})$ complex quadratic field, discriminant $d < 0$, 3-class group $\mathrm{Cl}_3(K) \simeq (3,3)$, $L_1, \ldots, L_4$ the unramified cyclic cubic extensions. If $\mathrm{Cl}_3(L_1) \simeq (27,9)$, $\mathrm{Cl}_3(L_j) \simeq (9,3)$, $j = 2,3,4$, and the 3-capitulation type $\varkappa_1(K)$ of K in $L_1, \ldots, L_4$ neither contains a total capitulation nor a 2-cycle, then the 3-class tower of K has length $\ell_3(K) = 3$, and the 3-class tower group $\mathrm{G}_3^\infty(K) = \mathrm{G}_3^3(K)$ is isomorphic to one of the three Schur σ-groups

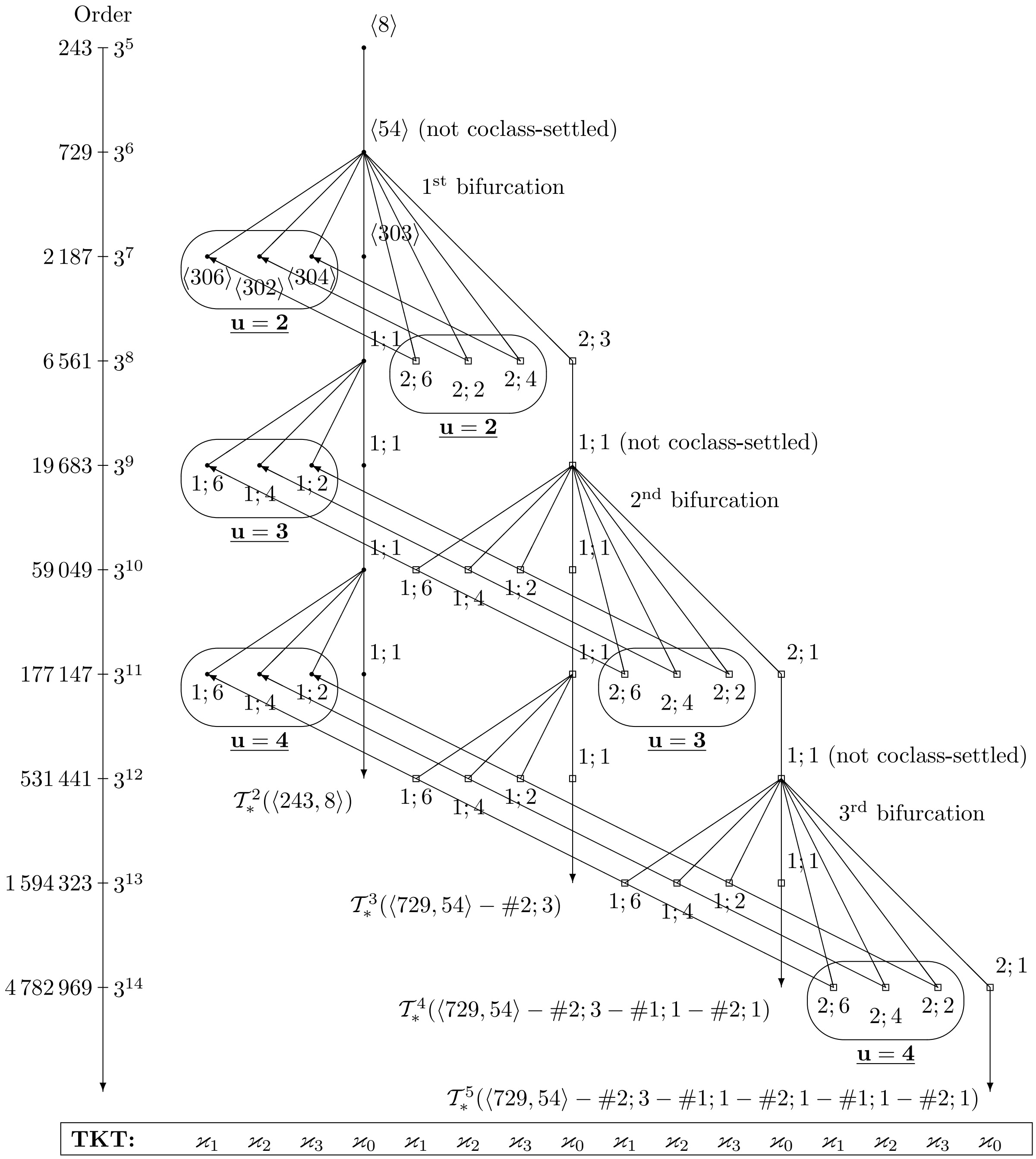
$$\langle 729, 54\rangle - \#2; 2|4|6.$$

[J1] M.R. Bush and D.C. Mayer,
3-*class field towers of exact length* 3,
J. Number Theory **147** (2015), 766–777,
DOI 10.1016/j.jnt.2014.08.010.

However, the 3-class tower groups in Theorem 1 are merely the beginning of periodic sequences of 3-class tower groups, as Theorem 2 will show.

Figure 1 shows the parameter u of various periodic sequences on the pruned tree $\mathcal{T}_*(\langle 243, 8\rangle)$.

FIGURE 1. Projections from non-metabelian 3-tower groups $G \in \mathcal{T}_*(\langle 243, 8\rangle)$ onto G/G''



Theorem 2. (D.C. Mayer, 2015)
Let $p = 3$, $K = \mathbb{Q}(\sqrt{d})$ complex quadratic field, discriminant $d < 0$, 3-class group $\mathrm{Cl}_3(K) \simeq (3,3)$, $L_1, \ldots, L_4$ the unramified cyclic cubic extensions, $K_1, \ldots, K_4$ their non-Galois cubic subfields.

Let $b \leq 9$ be a fixed upper bound, and $2 \leq u \leq b$. If the 3-class numbers of $K_1, \ldots, K_4$ are

$$h_3(K_1) = 3^u, \ h_3(K_j) = 3 \text{ for } 2 \leq j \leq 4,$$

and the 3-capitulation type $\varkappa_1(K)$ of K in $L_1, \ldots, L_4$ neither contains a total capitulation nor a 2-cycle, then the 3-class tower of K has length $\ell_3(K) = 3$, and either the 3-class tower group $\mathrm{G}_3^\infty(K) = \mathrm{G}_3^3(K)$ is isomorphic to one of the three Schur σ-groups

$$\langle 729, 54 \rangle \quad \overbrace{(-\#2; 1 - \#1; 1)^{u-2}}^{(u-2) \text{ primitive periods of length } 2} \quad -\#2; 2|4|6$$

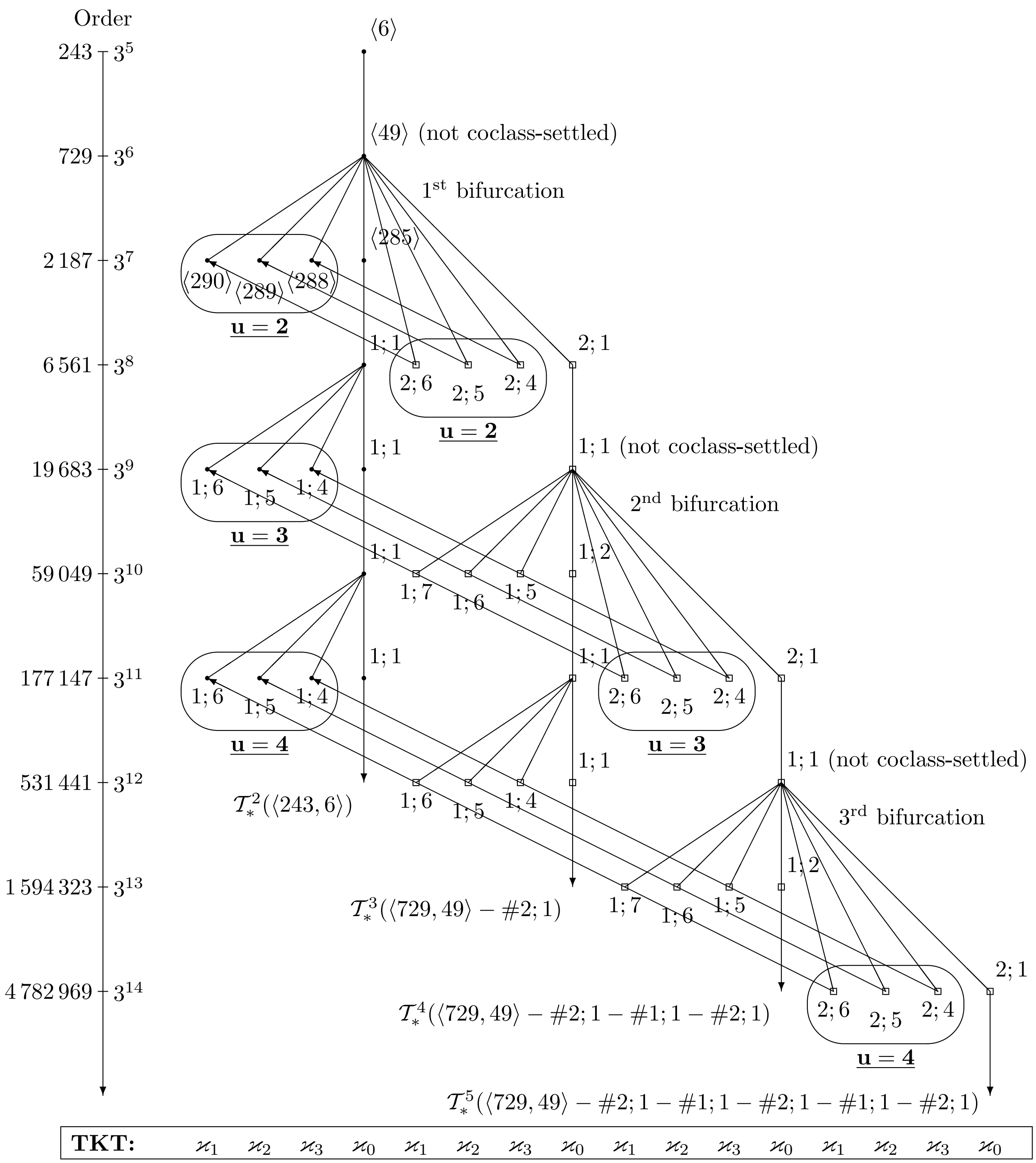
or it is isomorphic to one of the three Schur σ-groups

$$\langle 729, 49 \rangle \quad \overbrace{(-\#2; 1 - \#1; 1)^{u-2}}^{(u-2) \text{ primitive periods of length } 2} \quad -\#2; 4|5|6.$$

[T3] D.C. Mayer,
Periodic sequences of p-class tower groups,
J. Appl. Math. Phys. **3** (2015), 746–756,
DOI 10.4236/jamp.2015.37090.

Figure 2 shows the parameter u of various periodic sequences on the pruned tree $\mathcal{T}_*(\langle 243, 6\rangle)$.

FIGURE 2. Projections from non-metabelian 3-tower groups $G \in \mathcal{T}_*(\langle 243, 6\rangle)$ onto G/G''



New periodic sequences of length 2:

Firstly, we have **six** periodic sequences of 3-class tower groups $\mathrm{G}_3^\infty(K)$ of complex quadratic base fields $K = \mathbb{Q}(\sqrt{d})$, which must be **Schur** σ-groups: $(S_{k,4})_{k\geq 0}, (S_{k,5})_{k\geq 0}, (S_{k,6})_{k\geq 0}, (\bar{S}_{k,2})_{k\geq 0}, (\bar{S}_{k,4})_{k\geq 0}, (\bar{S}_{k,6})_{k\geq 0}$, where

$$S_{k,\ell} := \langle 729, 49\rangle \overbrace{(-\#2; 1 - \#1; 1)^k}^{k \text{ primitive periods}} -\#2; 4|5|6, \text{ resp.}$$

$$\bar{S}_{k,\ell} := \langle 729, 54\rangle \overbrace{(-\#2; 1 - \#1; 1)^k}^{k \text{ primitive periods}} -\#2; 2|4|6.$$

However, in both pruned descendant trees $\mathcal{T}_*(\langle 243,6\rangle)$ and $\mathcal{T}_*(\langle 243,8\rangle)$, there exist **two** further periodic sequences of finite 3-groups, which are of central importance for the tree structure:

$$(R_k)_{k\geq 0}, (\bar{R}_k)_{k\geq 0}, (B_k)_{k\geq 0}, (\bar{B}_k)_{k\geq 0},$$

where for each $k \geq 0$,
the **root** of the pruned coclass tree $\mathcal{T}_*^{k+2}$ is

$$R_k := \langle 243,6\rangle \overbrace{(-\#1;1-\#2;1)^k}^{k \text{ primitive periods}}, \text{ resp.}$$

$$\bar{R}_k := \langle 243,8\rangle \overbrace{(-\#1;1-\#2;1)^k}^{k \text{ primitive periods}},$$

and the immediate mainline descendant of the root with nuclear rank $\nu = 2$, which gives rise to the crucial **bifurcation**, is

$$B_k := \langle 243,6\rangle \overbrace{(-\#1;1-\#2;1)^k}^{k \text{ primitive periods}} -\#1;1, \text{ resp.}$$

$$\bar{B}_k := \langle 243,8\rangle \overbrace{(-\#1;1-\#2;1)^k}^{k \text{ primitive periods}} -\#1;1.$$

CHAPTER I.
THE GROUP THEORY
OF p-CLASS TOWER GROUPS

§ 1. The Artin Pattern

Let $p \geq 2$ be a prime number, G a pro-p group with commutator subgroup G' and finite abelianization G/G' of order p^v, $v \geq 1$.

Definition 1.1.

$\mathrm{Lyr}_n(G) := \{G' \leq H \trianglelefteq G \mid (G : H) = p^n\}$, $0 \leq n \leq v$, the $v+1$ *layers* of intermediate normal subgroups between G and G'.

$T_{G,H} : G/G' \to H/H'$ the *Artin transfer* [1] from G to H,

$\tau_n(G) := (H/H')_{H \in \mathrm{Lyr}_n(G)}$, $0 \leq n \leq v$, the components of the multi-layered *transfer target type* (TTT) $\tau(G) := [\tau_0(G); \ldots ; \tau_v(G)]$,

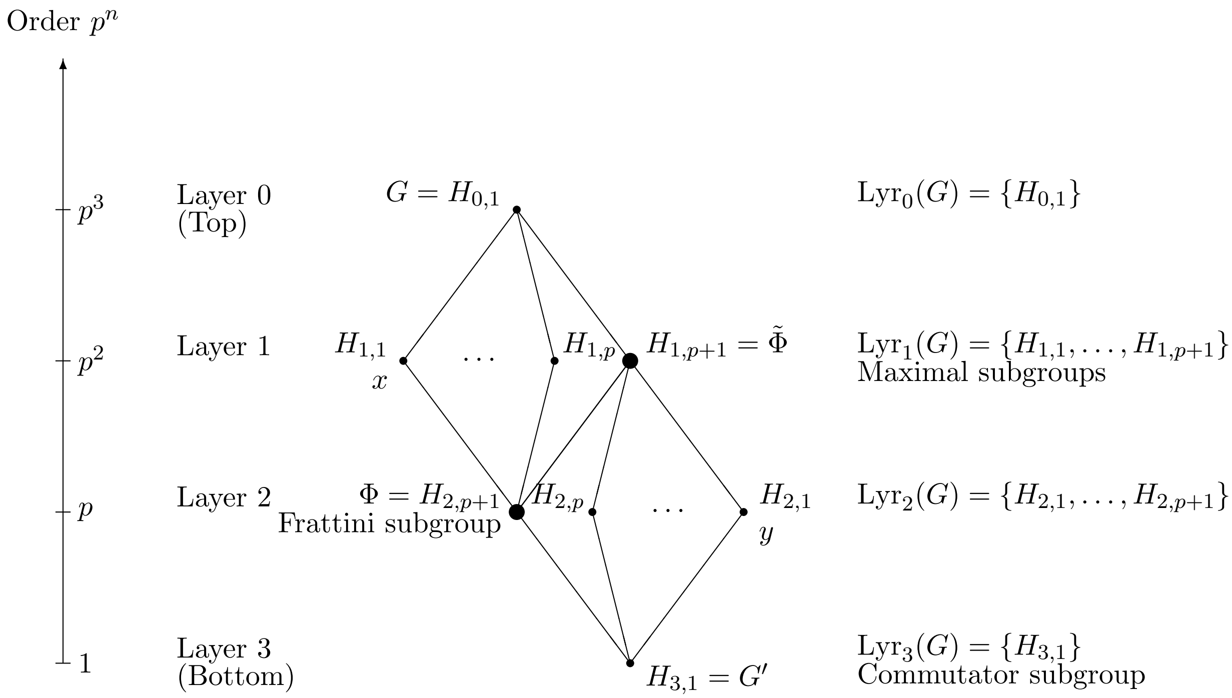
$\varkappa_n(G) := (\ker(T_{G,H}))_{H \in \mathrm{Lyr}_n(G)}$, $0 \leq n \leq v$, the components of the multi-layered *transfer kernel type* (TKT) $\varkappa(G) := [\varkappa_0(G); \ldots ; \varkappa_v(G)]$.

The pair $\mathrm{AP}(G) := (\tau(G), \varkappa(G))$ is called the **Artin pattern** of G.

[1] E. Artin, Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz, *Abh. Math. Sem. Univ. Hamburg* **7** (1929), 46–51.

Figure 1 shows a small non-trivial example of a multi-layered abelianization G/G'.

FIGURE 3. Layers of subgroups $G' \le H_{i,j} \le G$ for $G/G' = \langle x, y, G' \rangle \simeq (p^2, p)$



Definition 1.2.

(N. Boston, M.R. Bush, F. Hajir, 2011 [J4])

The *Index-p Abelianization Data* (IPAD) of G,

$$\tau^{(1)}(G) := [\tau_0(G); \tau_1(G)],$$

arises by restriction to the zeroth and first layer. It is a first order approximation of the TTT $\tau(G)$.

Example: In the situation of Figure 1, the IPAD of G is given by

$$\tau^{(1)}(G) = [G/G'; (H_{1,1}/H'_{1,1}, \dots, H_{1,p+1}/H'_{1,p+1})].$$

CHAPTER II. THE ARITHMETIC OF p-CLASS TOWER GROUPS

§ 2. Capitulation of p-Classes

Definition 2.1.
K a number field of p-class rank $r_p(K) = 2$,
$L_1, \ldots, L_{p+1}$
its unramified cyclic extension fields of degree p,
$j_i = j_{L_i|K} : \mathrm{Cl}_p(K) \to \mathrm{Cl}_p(L_i)$
the extension homomorphisms of p-classes.

The family $\varkappa_1(K) = (\ker(j_i))_{1 \le i \le p+1}$
is called the *p-capitulation type* of K [8], [MT2].

The family $\tau_1(K) = (\mathrm{Cl}_p(L_i))_{1 \le i \le p+1}$
is called the *p-class group type* of K [MT4].

Theorem 2.1. (E. Artin, 1929 [1])
The p-capitulation type $\varkappa_1(K)$, resp. p-class group type $\tau_1(K)$, of K coincides with the first layer TKT $\varkappa_1(G)$, resp. TTT $\tau_1(G)$, of the nth p-class group $G = \mathrm{G}_p^n(K)$, for any $2 \le n \le \infty$.

$$\begin{array}{ccccc} & & j_{L|K} & & \\ & \mathrm{Cl}_p(K) & \longrightarrow & \mathrm{Cl}_p(L) & \\ \text{Artin} & \updownarrow & & \updownarrow & \text{Artin} \\ \text{isomorphism} & G/G' & \longrightarrow & H/H' & \text{isomorphism} \\ & & T_{G,H} & & \end{array}$$

§ 3. Relation Rank of p-Class Tower Groups

Theorem 3.1. (I.R. Shafarevich, 1964 [9])
$p \geq 2$ prime number, K number field with signature (r_1, r_2) and torsion free unit rank $r = r_1 + r_2 - 1$,
S finite set of places of K not divisible by p,
ζ primitive pth root of unity,
$G := \mathrm{G}_{S,p}^{\infty}(K) = \mathrm{Gal}(\mathrm{F}_{S,p}^{\infty}(K)|K)$ the Galois group of the maximal pro-p extension $\mathrm{F}_{S,p}^{\infty}(K)$ of K which is unramified outside of S,
$d_1 := \dim_{\mathbb{F}_p} \mathrm{H}^1(G, \mathbb{F}_p)$ the *generator rank* of G,
$d_2 := \dim_{\mathbb{F}_p} \mathrm{H}^2(G, \mathbb{F}_p)$ the *relation rank* of G. Then

$$d_1 \leq d_2 \leq \begin{cases} d_1 + r & \text{if } S \neq \emptyset \text{ or } \zeta \notin K, \\ d_1 + 1 & \text{if } S = \emptyset \text{ and } \zeta \in K. \end{cases}$$

Corollary 3.1. $K = \mathbb{Q}(\sqrt{d})$ quadratic field with discriminant d and $S = \emptyset$,
$G := \mathrm{G}_p^{\infty}(K) = \mathrm{Gal}(\mathrm{F}_p^{\infty}(K)|K)$ the Galois group of the maximal unramified pro-p extension $\mathrm{F}_p^{\infty}(K)$ of K, i.e., the *p-class tower group* of K. Then

$$\begin{cases} d_2 = d_1 & \text{if } (d < 0 \text{ and } p \geq 3 \text{ odd}), \\ d_1 \leq d_2 \leq d_1 + 1 & \text{if either } (d < 0 \text{ and } p = 2) \text{ or } d > 0. \end{cases}$$

§ 4. Three-Stage p-Class Towers

Theorem 4.1. (=Thm.1) (Bush, Mayer, 2012)
Let $p = 3$ and $K = \mathbb{Q}(\sqrt{d})$ a complex quadratic field with discriminant $d < 0$ and IPAD of 1^{st} order

$$\tau^{(1)}(K) = [\tau_0(K); \tau_1(K)] = [1^2; (32, (21)^3)].$$

If the 3-capitulation type $\varkappa_1(K)$
neither contains a total capitulation nor a 2-cycle, then $\mathrm{G}_3^\infty(K) \simeq \langle 729, 54\rangle - \#2; 2|4|6$, one of three Schur σ-groups of derived length 3, and $\ell_3(K) = 3$.

Theorem 4.2. (=Thm.2) (D.C. Mayer, 2015 [T3])
Let $p = 3$ and $K = \mathbb{Q}(\sqrt{d})$ a complex quadratic field with discriminant $d < 0$ and IPAD of 1^{st} order

$$\tau^{(1)}(K) = [\tau_0(K); \tau_1(K)] = [1^2; ((u{+}1, u), 1^3, (21)^2)],$$

resp.

$$\tau^{(1)}(K) = [\tau_0(K); \tau_1(K)] = [1^2; ((u + 1, u), (21)^3)],$$

where $2 \le u \le 8$. If the 3-capitulation type $\varkappa_1(K)$ neither contains a total capitulation nor a 2-cycle, then

$$\mathrm{G}_3^\infty(K) \simeq \langle 729, 49\rangle(-\#2; 1-\#1; 1)^{u-2} - \#2; 4|5|6,$$

resp.

$$\mathrm{G}_3^\infty(K) \simeq \langle 729, 54\rangle(-\#2; 1-\#1; 1)^{u-2} - \#2; 2|4|6,$$

one of six Schur σ-groups of derived length 3, and thus $\ell_3(K) = 3$.

Figure 4 shows the parameter u of Thm. 4.2 on the coclass tree with root $\langle 243, 6\rangle$.

FIGURE 4. Parameter u of metabelian quotients $G/G'' \in \mathcal{T}^2(\langle 243,6\rangle)$ of G

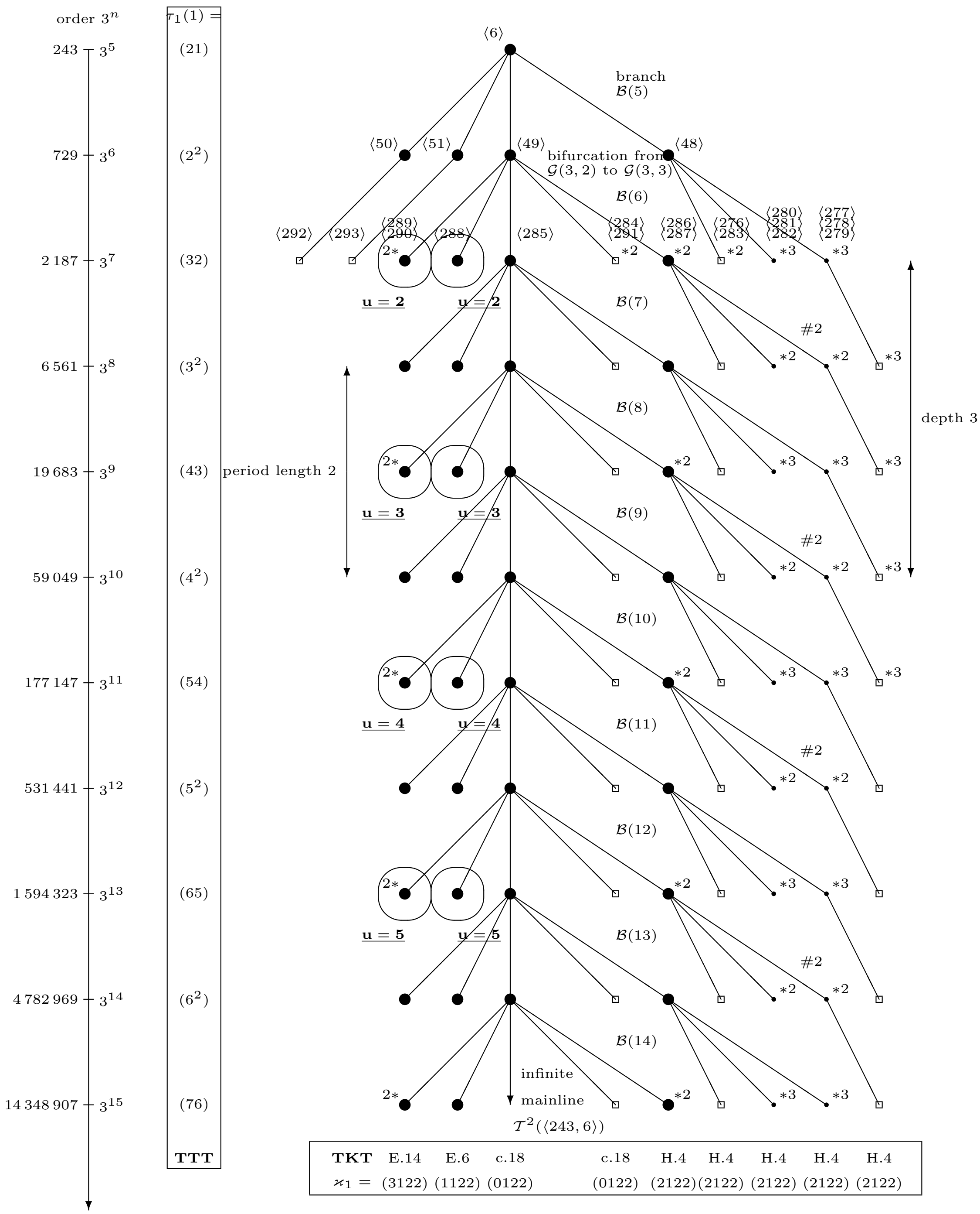
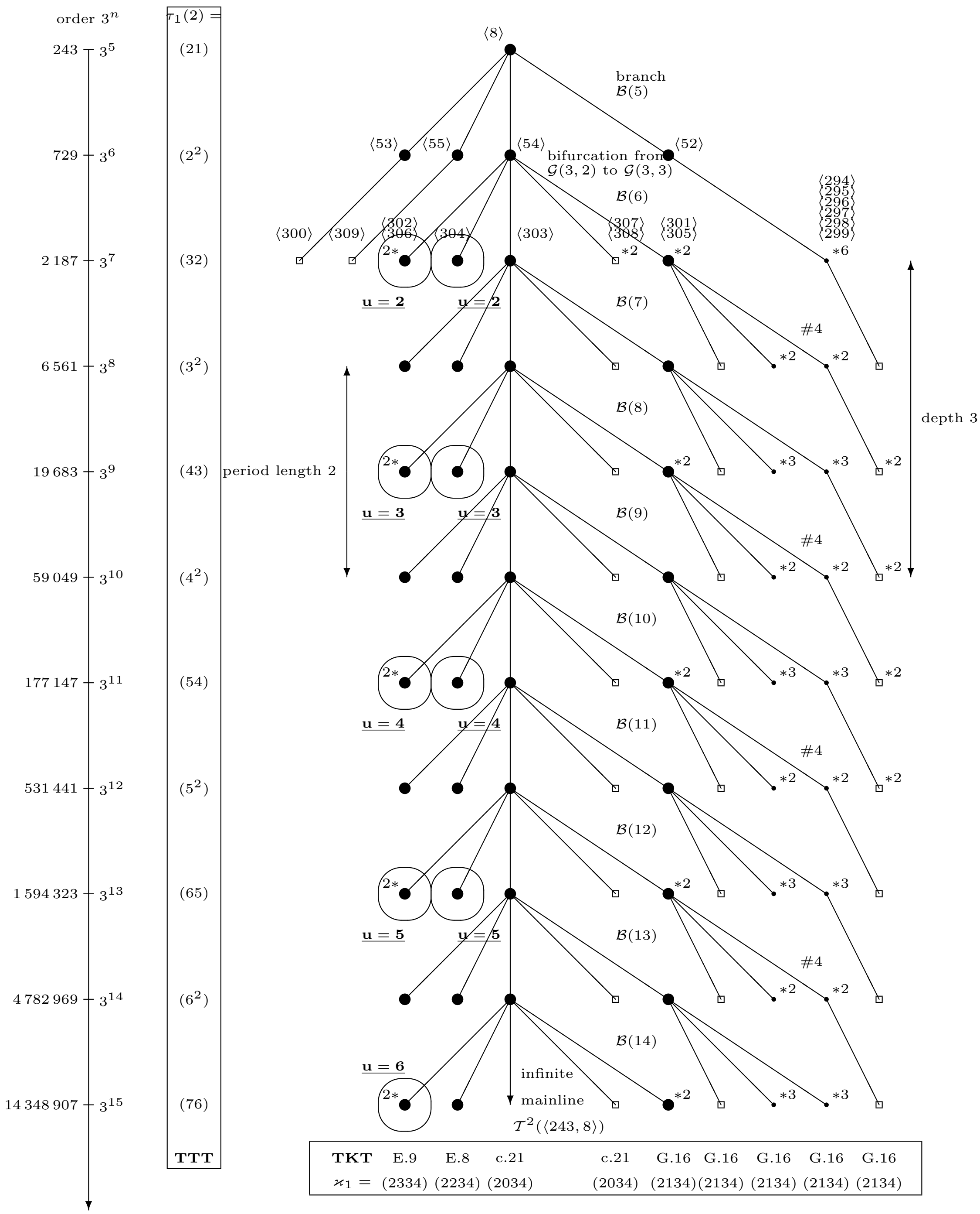


Figure 5 shows the parameter u of Thm. 4.2 on the coclass tree with root $\langle 243, 8\rangle$.

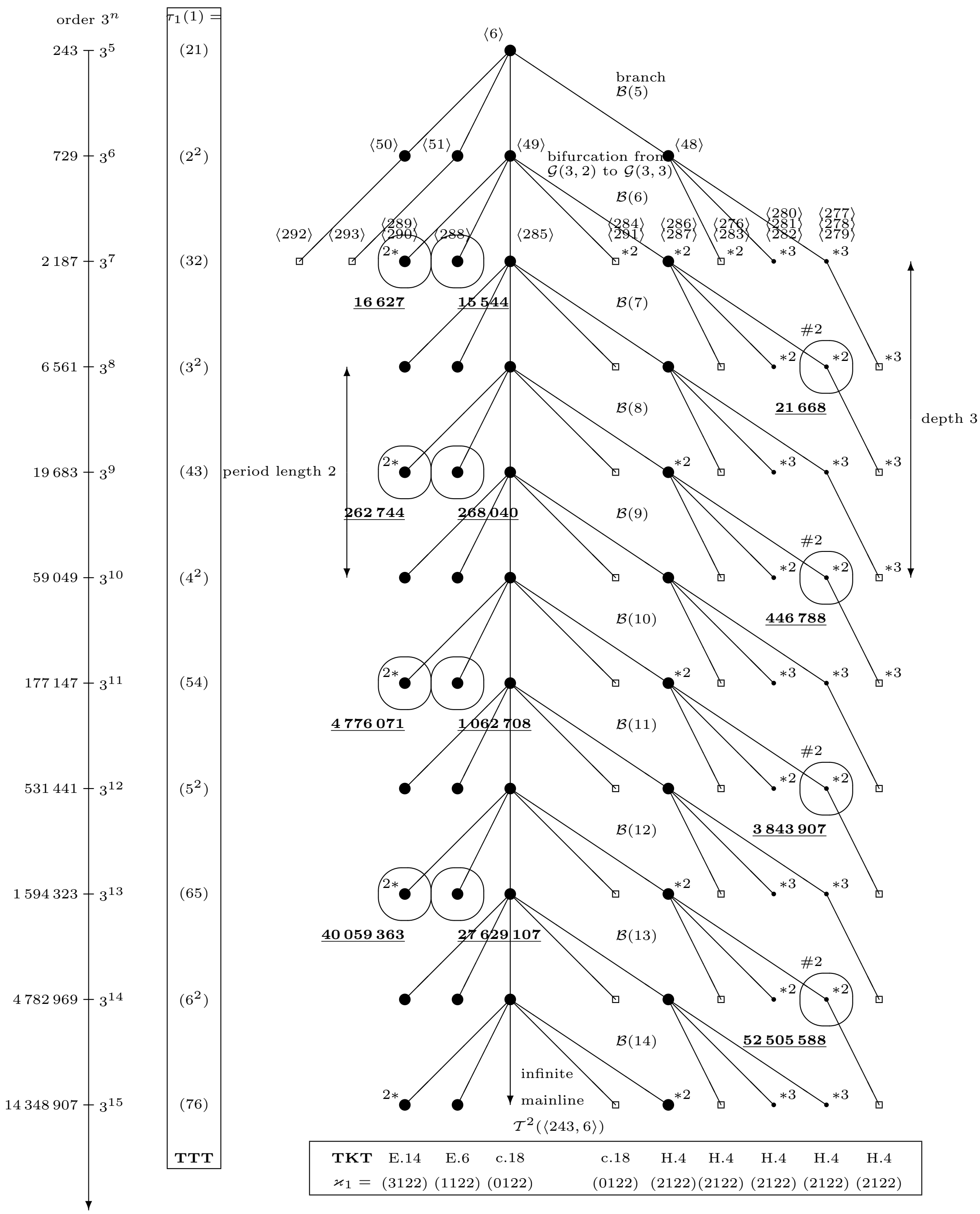
FIGURE 5. Parameter u of metabelian quotients $G/G'' \in \mathcal{T}^2(\langle 243, 8\rangle)$ of G



TKT	E.9	E.8	c.21	c.21	G.16	G.16	G.16	G.16	G.16
$\varkappa_1 =$	(2334)	(2234)	(2034)	(2034)	(2134)	(2134)	(2134)	(2134)	(2134)

Figure 6 shows the population of the coclass tree with root $\langle 243, 6\rangle$ by $|d|$ of $K = \mathbb{Q}(\sqrt{d})$, $d < 0$.

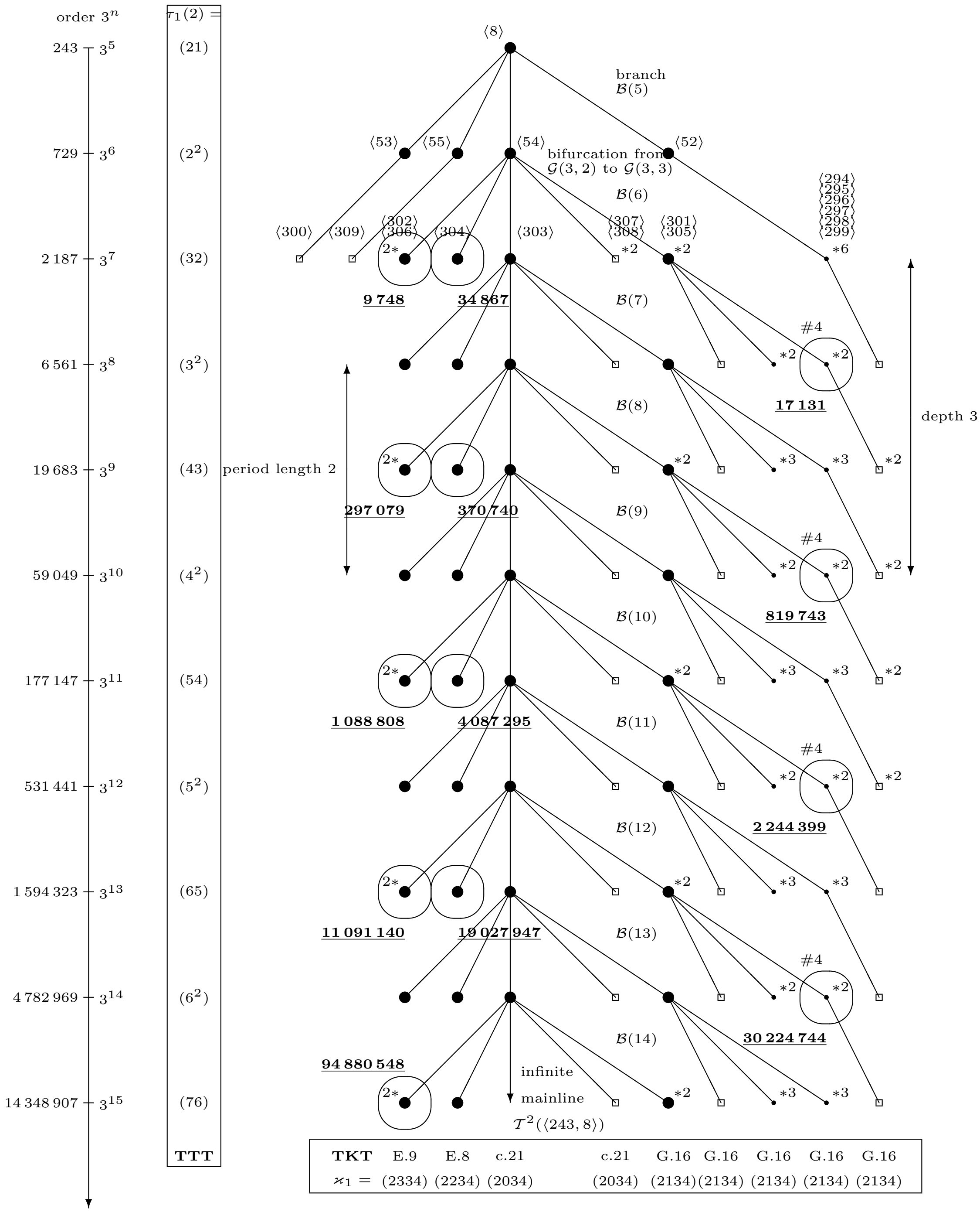
FIGURE 6. Population of metabelian quotients $G/G'' \in \mathcal{T}^2(\langle 243, 6\rangle)$ of G



TKT	E.14	E.6	c.18	c.18	H.4	H.4	H.4	H.4	H.4
$\varkappa_1 =$	(3122)	(1122)	(0122)	(0122)	(2122)	(2122)	(2122)	(2122)	(2122)

Figure 7 shows the population of the coclass tree with root $\langle 243, 8\rangle$ by $|d|$ of $K = \mathbb{Q}(\sqrt{d})$, $d < 0$.

FIGURE 7. Population of metabelian quotients $G/G'' \in \mathcal{T}^2(\langle 243, 8\rangle)$ of G



Theorem 4.3. (D.C. Mayer, 2015)
Let $p = 3$ and K a number field with 3-class group $\mathrm{Cl}_3(K)$ of type $(3,3)$. Suppose $L_1, \ldots, L_4$ are the unramified cyclic cubic extensions of K within the first Hilbert 3-class field $\mathrm{F}_3^1(K)$. Let $\tau^{(1)}(L_i) = [\tau_0(L_i); \tau_1(L_i)]$ be the IPAD of L_i, for $1 \le i \le 4$.

If the 3-capitulation type $\varkappa_1(K)$
neither contains a total capitulation nor a 2-cycle, and K has the iterated IPAD of 2^{nd} order

$$\tau^{(2)}(K) = [\tau_0(K); (\tau^{(1)}(L_1), \ldots, \tau^{(1)}(L_4))],$$

where $\tau_0(K) = 1^2$, $\tau^{(1)}(L_1) = [32; (2^2 1, (31^2)^3)]$, then

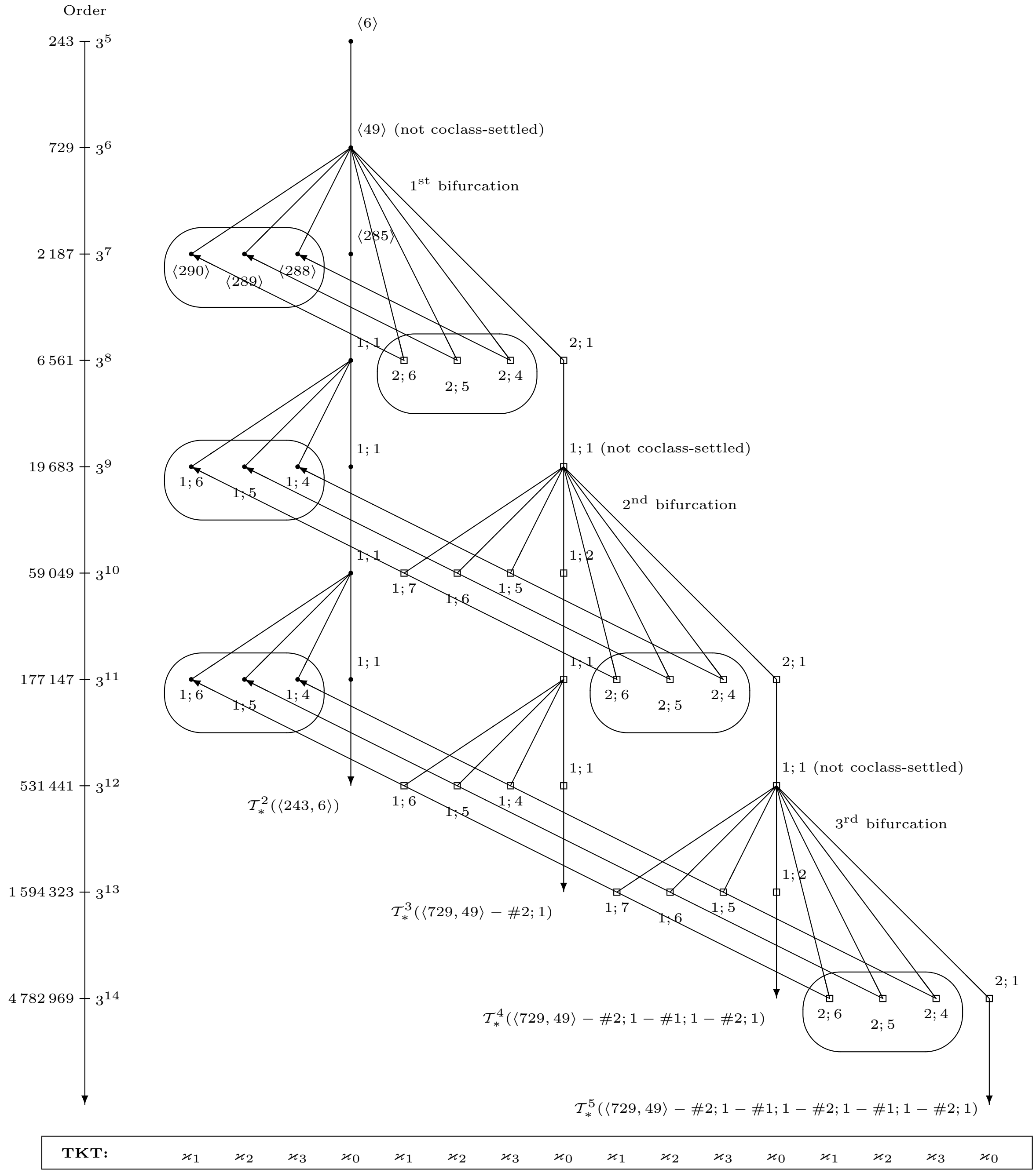
1.
$\tau^{(1)}(L_i) = [21; (2^2 1, (\mathbf{2}1)^3)]$, for $2 \le i \le 4$, $\Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 729, 54\rangle - \#\mathbf{1}; 2|4|6 = \langle 2187, 302|304|306\rangle$, one of three metabelian unbalanced σ-groups, and $\ell_3(K) = 2$,

2.
$\tau^{(1)}(L_i) = [21; (2^2 1, (\mathbf{3}1)^3)]$, for $2 \le i \le 4$, $\Longrightarrow$
$\mathrm{G}_3^\infty(K) \simeq \langle 729, 54\rangle - \#\mathbf{2}; 2|4|6$, one of three Schur σ-groups of derived length 3, and $\ell_3(K) = 3$.

[T2] D.C. Mayer,
Index-p abelianization data of
p-class tower groups,
Adv. Pure Math. **5** (2015), no. 5, 286–313.

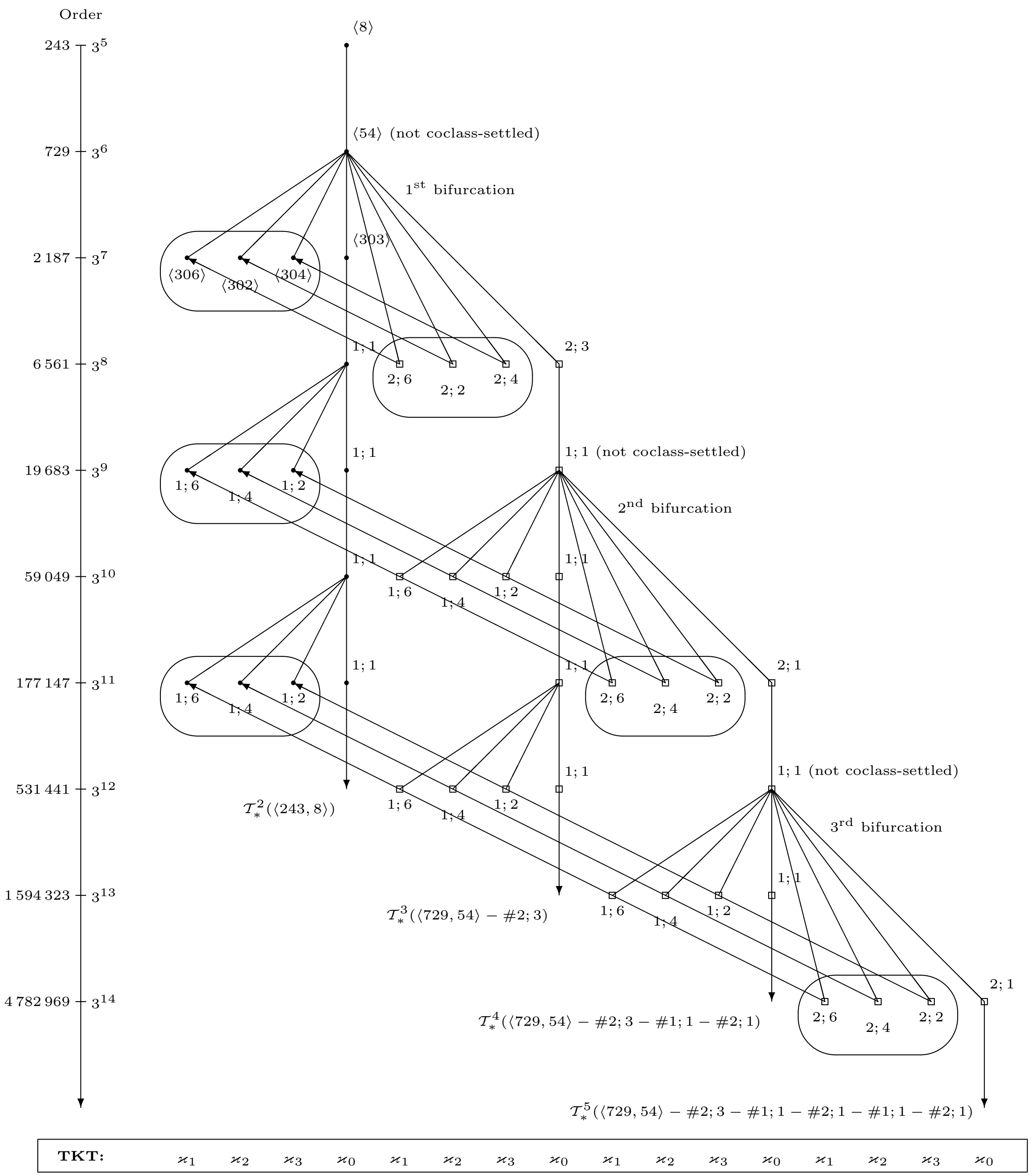
The tree in Figure 8 contains the three non-metabelian Schur σ-groups $G = \langle 729, 49\rangle - \#2; 4|5|6$ with $d_2 = d_1 = 2$ and G/G' of type $(3, 3)$.

FIGURE 8. Projections from non-metabelian 3-tower groups $G \in \mathcal{T}_*(\langle 243, 6\rangle)$ onto G/G''



The tree in Figure 9 contains the three non-metabelian Schur σ-groups $G = \langle 729, 54\rangle - \#2; 2|4|6$ with $d_2 = d_1 = 2$ and G/G' of type $(3, 3)$.

FIGURE 9. Projections from non-metabelian 3-tower groups $G \in \mathcal{T}_*(\langle 243, 8\rangle)$ onto G/G''



§ 5. Two-Stage 2-Class Towers

Theorem 5.1. (Azizi, Zekhnini, Taous, Mayer)
Two complex quadratic fields and 2-class numbers,
$p_1 \equiv 1 \pmod 8$ prime,
$k_1 = \mathbb{Q}(\sqrt{-p_1})$, $h_2(k_1) = 2^{m+1}$, $(m \ge 1)$,
$p_2 \equiv 5 \pmod 8$ prime, $q \equiv 3 \pmod 4$ prime,
$k_2 = \mathbb{Q}(\sqrt{-p_2 q})$, $h_2(k_2) = 2^n$, $(n \ge 1)$.
A bicyclic biquadratic field and its 2-class tower,
$\left(\frac{p_1}{p_2}\right) = -1$, $\left(\frac{p_1}{q}\right) = -1$,
$K = \mathbb{Q}(\sqrt{-1}, \sqrt{p_1 p_2 q})$, $\mathfrak{G} := \mathrm{G}_2^\infty(K)$.

1. If $\left(\frac{p_2}{q}\right) = -1$, then $\mathfrak{G} \simeq \langle 32, 35\rangle(-\#1;1)^{m-1}$ on the mainline of the coclass tree $\mathcal{T}^3(\langle 32, 35\rangle)$.

2. If $\left(\frac{p_2}{q}\right) = +1$ and $m \ge n$, then

$$\mathfrak{G} \simeq \langle 32, 34\rangle(-\#2;1)^{n-2} - \#2;2(-\#1;1)^{m-n}$$

on the mainline of the coclass tree

$$\mathcal{T}^{n+2}(\langle 32, 34\rangle(-\#2;1)^{n-2} - \#2;2).$$

3. If $\left(\frac{p_2}{q}\right) = +1$ and $n > m$, then

$$\mathfrak{G} \simeq \langle 32, 34\rangle(-\#2;1)^{m-1}(-\#1;1)^{n-m-1} - \#1;2$$

on the periodic coclass sequence with a permutation as the first layer transfer kernel type $\varkappa_1$ of the coclass tree

$$\mathcal{T}^{m+2}(\langle 32, 34\rangle(-\#2;1)^{m-1}).$$

New periodic sequence of length 1:

In the pruned descendant tree $\mathcal{T}_*(\langle 8,5\rangle)$, there exists a periodic sequence of finite 2-groups, which is of central importance for the tree structure:

$$(\delta^k(G))_{k\geq 0},$$

where for each $k\geq 0$,
the root of a coclass tree $\mathcal{T}^{k+3}$ with nuclear rank $\nu=2$, which gives rise to the crucial bifurcation, is

$$\delta^k(G) := G \quad \overbrace{(-\#2;1)^k}^{k\text{ primitive periods}} \quad \text{with } G := \langle 32,34\rangle.$$

[J3] A. Azizi, A. Zekhnini and M. Taous,
Coclass of $\mathrm{Gal}(k_2^{(2)}|k)$ *for some fields*
$k = \mathbb{Q}\left(\sqrt{p_1p_2q},\sqrt{-1}\right)$ *with* 2*-class groups of type* $(2,2,2)$, to appear in J. Algebra Appl., 2015.

Figure 10, resp. 11, shows the distribution of parameters (m,n), resp. minimal radicands $d=p_1p_2q$, on the pruned tree $\mathcal{T}_*(\langle 8|5\rangle)$.

FIGURE 10. Parameters (m,n) of metabelian 2-tower groups $\mathfrak{G} \in \mathcal{T}_*(\langle 32,34|35\rangle)$

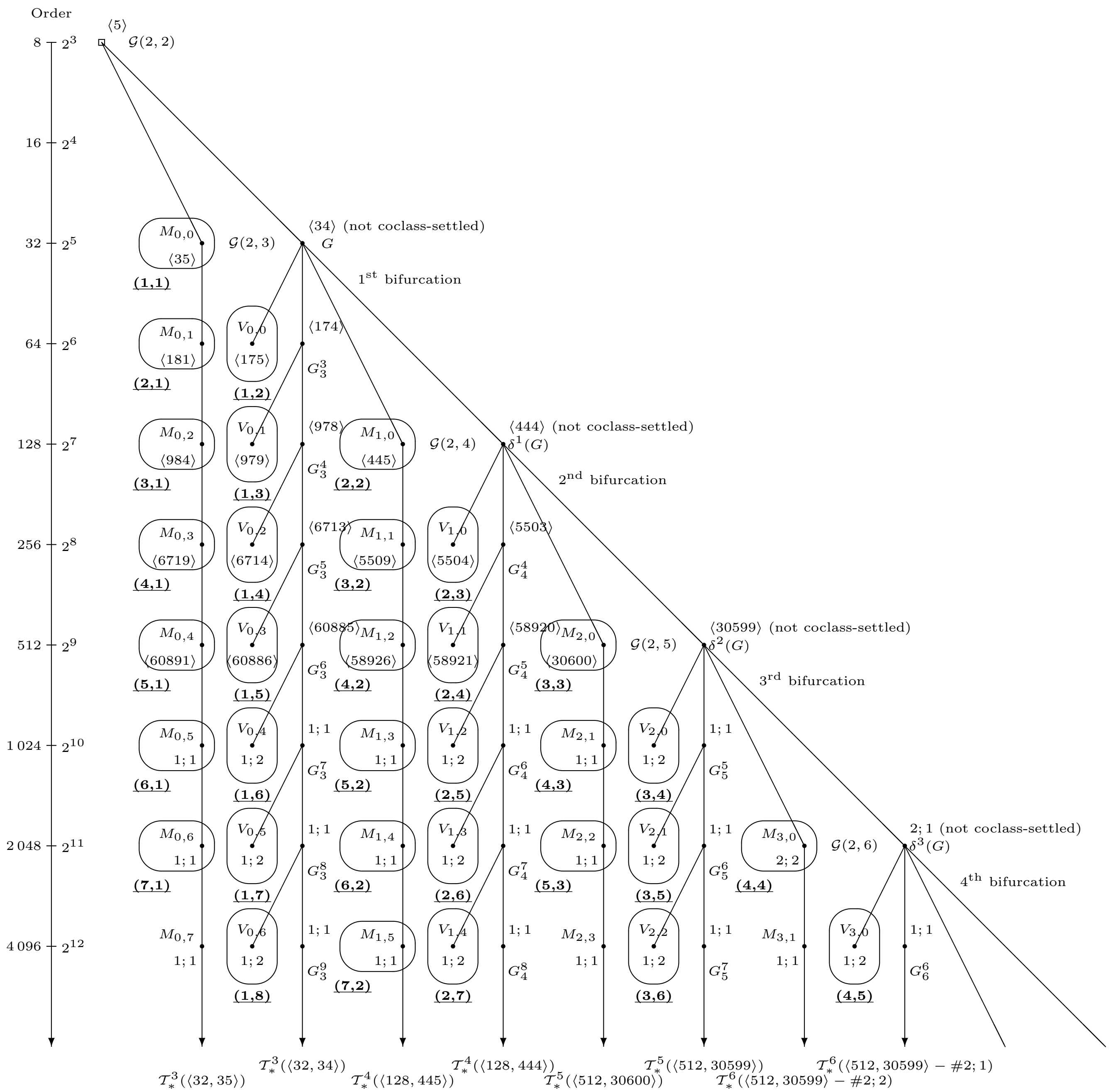
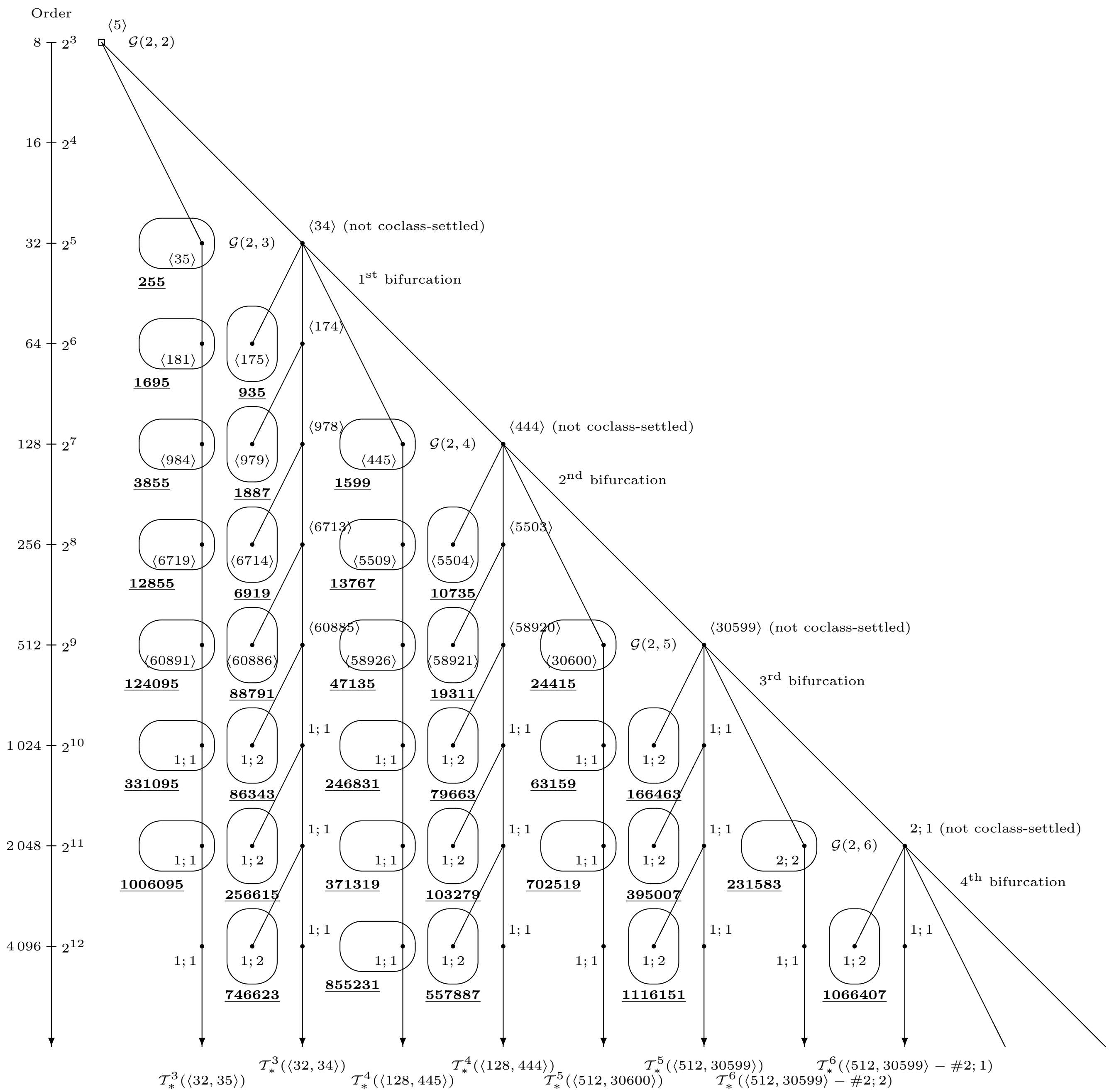


FIGURE 11. Population of metabelian 2-tower groups $\mathfrak{G} \in \mathcal{T}_*(\langle 32, 34|35\rangle)$



Top-Recent Trilogy.

[T1] D.C. Mayer,
Periodic bifurcations in
descendant trees of finite p-groups,
Adv. Pure Math. **5** (2015), no. 4, 162–195,
DOI 10.4236/apm.2015.54020,
Special Issue on Group Theory.

[T2] D.C. Mayer,
Index-p abelianization data of
p-class tower groups,
Adv. Pure Math. **5** (2015), no. 5, 286–313,
DOI 10.4236/apm.2015.55029,
Special Issue on Number Theory
and Cryptography.
(29th Journées Arithmétiques 2015,
University of Debrecen, Hungary, 09 July 2015.)

[T3] D.C. Mayer,
Periodic sequences of p-class tower groups,
J. Appl. Math. Phys. **3** (2015), 746–756,
DOI 10.4236/jamp.2015.37090.
(International Conference on
Groups and Algebras 2015,
Shanghai, China, 21 July 2015.)

Most Recent Joint Works.

[J1] M.R. Bush and D.C. Mayer,
3-class field towers of exact length 3,
J. Number Theory **147** (2015), 766–777,
DOI 10.1016/j.jnt.2014.08.010.

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