FINAL REPORT ON THE STAND-ALONE PROJECT P 26008-N25, "TOWERS OF p-CLASS FIELDS OVER ALGEBRAIC NUMBER FIELDS"

DANIEL C. MAYER

1. Report on research work. The following final report 2016 concludes my stand-alone project P 26008-N25, as described in the proposal [13]. I refer to various URLs in the world wide web, in particular to my "Website for Principal Investigators of the FWF" [14]. The project started with my presentation "Finite 3-groups as viewed from class field theory"[15] at the international group theory conference "Groups St. Andrews" on 11 August 2013, where I predicted the beginning of a new era in obtaining results on p-class towers with the aid of periodic bifurcations in descendant trees of finite p-groups and Schur covers of metabelian p-groups (now replaced by the more general concept of Shafarevich covers). The project ended with a summary of my current view of p-class towers from a high level scope, which I gained during the past three years, presented to the scientific community in my invited keynote "Recent Progress in Determining p-Class Field Towers" [33], at the "1st International Colloquium of Algebra, Number Theory, Cryptography and Information Security" in Taza, Morocco, on 12 November 2016.

1.1. Information on the development of the research project.

1.1.1. *Overall scientific concept and goals.* For an assigned prime number p, the Hilbert p-class field tower $\mathrm{F}_p^\infty K := \bigcup_{n\ge 0} \mathrm{F}_p^n K$ is the maximal unramified pro-p extension of an algebraic number field K. It consists of successively nested maximal unramified abelian p-extensions $\mathrm{F}_p^n K := \mathrm{F}_p^1\left(\mathrm{F}_p^{n-1} K\right)$, for $n \ge 1$, of the base field $\mathrm{F}_p^0 K := K$. Every stage $\mathrm{F}_p^n K$ of the tower is associated with a finite p-group $\mathrm{G}_p^n K := \mathrm{Gal}\left(\mathrm{F}_p^n K/K\right)$, which is called the nth p-class group of K. The principal goal of this project was the identification of the p-tower group $\mathrm{G}_p^\infty K := \mathrm{Gal}\left(\mathrm{F}_p^\infty K/K\right) \simeq \varprojlim_{n\ge 0} \mathrm{G}_p^n K$ by an *explicit pro-p presentation*, for selected kinds of base fields K, specified by their signature (r_1, r_2) and p-class group $\mathrm{Cl}_p K$ [13, § 1.6, p. 16].

1.1.2. *Fundamental changes in the research orientation.* During the development of the project, I *firstly* replaced the original intention of my proposal, to determine the metabelian two-stage approximation $\mathfrak{M} := \mathrm{G}_p^2 K \simeq G/G''$ of the p-tower group $G := \mathrm{G}_p^\infty K$ and the length

Date: November 30, 2016.

2000 *Mathematics Subject Classification.* Primary 11R37, 11R29, 11R11, 11R16, 11R20; Secondary 20D15, 20E18, 20E22, 20F05, 20F12, 20F14, 20–04.

Key words and phrases. Hilbert p-class field towers, maximal unramified pro-p extensions, second and higher p-class groups, principalization of p-classes, quadratic fields, cubic fields, quartic fields, quintic fields, dihedral fields, Frobenius fields; finite p-groups, descendant trees, pro-p groups, coclass trees, abelian quotient invariants, Artin transfer pattern, p-group generation algorithm, p-covering group, p-multiplier, nucleus, nuclear rank, multifurcation, generator rank, relation rank, Shafarevich cover, balanced presentation, Schur σ-groups, Galois action, polycyclic presentations, commutator calculus, central series, annihilator ideals.

$\ell_p K := \mathrm{dl}(G)$ of the p-class tower, with the more rigorous postulate that the tower is to be considered as *known* only if an explicit pro-p presentation $1 \to \mathfrak{R} \to \mathfrak{F} \to G \to 1$ of G is given [23, Abstract, p. 21, and § 10, p. 54]. The pro-p presentation of G admits the computation of polycyclic power-commutator presentations of all higher p-class groups $\mathrm{G}_p^n K \simeq G/G^{(n)}$ by forming the nth derived quotient of G, for $n \geq 1$. Further, it determines all crucial invariants of G, like the Artin pattern $\mathrm{AP}(G)$, the relation rank $d_2 G$, and the behavior of G under the action of the absolute Galois group $\mathrm{Gal}(K/\mathbb{Q})$. *Secondly*, I followed suggestions of the referees, who predicted regularization phenomena of p-capitulation types, on the one hand for primes $p \geq 5$, and on the other hand for normal base fields K with bigger Galois groups $\mathrm{Gal}(K/\mathbb{Q})$ than quadratic fields $K = \mathbb{Q}(\sqrt{d})$, and expressed the desire for general theorems on p-capitulation in the style of Miyake [37, 38], Arrigoni [1], and Suzuki [40]. The effect of these changes was an achievement of much deeper results than expected originally, for general p-groups [32, §§ 1.6–1.7], [33, § 5, pp. 11-14] as well as for particular kinds of base fields [23]. I proved an impressive confirmation of the predicted regularization by the Galois action [13, § 1.1.3] of cyclic groups $\mathrm{Gal}(K/\mathbb{Q}) = \langle\sigma\rangle$ of order coprime to p [33, § 10, pp. 29–33], and for second p-class groups $\mathfrak{M}$ of class $\mathrm{cl}(\mathfrak{M}) < p$ [32, § 7], which are *regular* in the sense of P. Hall. However, for bigger nilpotency class of $\mathfrak{M}$, we cannot speak about regularization and many mysterious 5-capitulation types are still waiting for clarification. For instance, the complex cyclic quartic field $K = \mathbb{Q}(\sqrt{a(d + b\sqrt{d})})$ with $a = -3 \cdot 1013$, $b = 1$, $d = 5$, possesses a very unusual IPAD $\tau_1 K = (21^3, 1^5, 21^3, 21^3, 1^5, 1^5)$ and IPOD $\varkappa_1 K = (1, 4, 3, 2, 4, 4)$ with two fixed points, 1 and 3, a 2-cycle $(2, 4)$, and two repetitions of 4, for which $\mathfrak{M} = \mathrm{G}_p^2 K$ is incomputable (due to unmanageable nuclear ranks) and thus completely unknown.

1.2. Most important results and their significance.

1.2.1. *Development of new methods.* In my proposal, I suggested to exploit the information provided by the p-class *transfer* homomorphisms $T_{K,L} : \mathrm{Cl}_p K \to \mathrm{Cl}_p L$ of all abelian unramified p-extensions L/K for determining the second p-class group $\mathfrak{M} = \mathrm{G}_p^2 K$ of K. It turned out that a separation of the extensions L/K in various *layers* is useful. If $\mathrm{Cl}_p K$ is of order p^t with $t \geq 0$, then we have $t+1$ layers $\mathrm{Lyr}_n K := \{K \leq L \leq \mathrm{F}_p^1 K \mid [L : K] = p^n\}$, $0 \leq n \leq t$, of abelian unramified p-extensions L/K, and for each of them we collect the targets $\tau_n K := (\mathrm{Cl}_p L)_{L \in \mathrm{Lyr}_n K}$ and the kernels $\varkappa_n K := (\ker T_{K,L})_{L \in \mathrm{Lyr}_n K}$ of the transfers in families. The *abelian Artin pattern* $\mathrm{AP}(K) := (\tau(K), \varkappa(K))$ of K consists of data for all layers, the *transfer target type* (TTT) $\tau(K) := [\tau_0 K; \ldots; \tau_t K]$ and the *transfer kernel type* (TKT) $\varkappa(K) := [\varkappa_0 K; \ldots; \varkappa_t K]$. Dropping the condition $L \leq \mathrm{F}_p^1 K$, we obtain the *total Artin pattern* $\mathrm{AP}_{\mathrm{tot}}(K) := (\tau_{\mathrm{tot}}(K), \varkappa_{\mathrm{tot}}(K))$ of K. However, for identifying the metabelianization $\mathfrak{M} = \mathrm{G}_p^2 K$ of $G = \mathrm{G}_p^\infty K$, the first layer $\tau_1 K$ of the TTT suffices already. This is due to the **Polarization Principle** [33, § 6, p. 15], which generally ensures that a few distinguished "polarized" components of $\tau_1 K$ determine the coclass $r := \mathrm{cc}(\mathfrak{M})$ and the nilpotency class $c := \mathrm{cl}(\mathfrak{M})$ of $\mathfrak{M}$. Together with the finiteness theorems of coclass theory, the polarization principle asserts that there is only a finite batch of metabelian p-groups $\mathfrak{M}$ having assigned *index-p abelianization data* (IPAD) $\tau^{(1)}K := [\tau_0 K; \tau_1 K]$ of K. The corresponding *index-p obstruction data* (IPOD) $\varkappa^{(1)}K := [\varkappa_0 K; \varkappa_1 K]$ of K and the remaining TTT layers $[\tau_2 K; \ldots; \tau_t K]$ of K can be used for narrowing down the finite batch further. I succeeded in giving a rigorous proof of the polarization principle for a p-class group $\mathrm{Cl}_p K$ of type (p,p) by showing that there exist two distinguished TTT components, a polarization $\mathrm{Cl}_p L_1$ with order p^{c-k}, $0 \le k \le 1$, and a co-polarization $\mathrm{Cl}_p L_2$ with order p^{r+1} [34, Thm. 5.1–5.2]. Together with my coworkers, we also were able to show for a 2-class group $\mathrm{Cl}_2 K$ of type $(2,2,2)$ that there exists a unique distinguished TTT component $\mathrm{Cl}_2 L_3$, with L_3 lying within the absolute 2-genus field $(K|\mathbb{Q})^*$ of K, which decomposes in two direct summands $(2^c, 2^{r-1})$ [20, Eqn. (4.2)–(4.4), pp. 749–750]. For other cases, the polarization principle is still a conjecture. An important break-off condition for the p-group generation algorithm is provided by the **Monotony Principle** [29, Thm. 3.1], [32, § 7, Thm. 1.21], [33, § 5, Thm. M, p. 14].

1.2.2. *Development or disproof of most important hypotheses.* Many new hypotheses were developed in the project: the mainline principle, the polarization principle (§ 1.2.1), and the stage separation conjecture (§ 1.2.3). They have been proved in particular cases but their general claims are still open problems. However, a single old hypothesis could be disproved with the aid of the coclass theorems. The erroneous old hypothesis was briefly stated in the abstract of my preprint on the strategy of "Pattern recognition via Artin transfers" [14, Project publications, (b.1)]. It was my fault to assume that the projective limit $L := \varprojlim_{i\ge 0} M_i$ of the mainline of a coclass tree $\mathcal{T}^r(M_0)$ with $r \ge 1$ could occur as Galois group $G = \mathrm{G}_p^\infty K$ of an infinite p-class tower $\mathrm{F}_p^\infty K/K$. In this case, G would be an infinite pro-p group of finite coclass r, in contradiction to the coclass theorems. Thus, *an infinite p-tower group G must necessarily have infinite coclass* $\mathrm{cc}(G) = \infty$. Purely group theoretic, we have:

Mainline Principle. Let $\mathcal{T}$ be a coclass tree of finite p-groups with generator rank n. Suppose that $\mathfrak{M}$ is a metabelian vertex of $\mathcal{T}$. Then there exist suitably normalized generators $x_1, \ldots, x_n$ and integers $e_1, \ldots, e_n$ such that a parametrized polycyclic power-commutator-presentation of $\mathfrak{M}$, with the nilpotency class $c = \mathrm{cl}(\mathfrak{M})$ as its parameter, is given by

$$\begin{aligned} \mathfrak{M} = \langle x_1, \ldots, x_n, s_2, \ldots, s_c \mid & s_2 = [x_2, x_1], s_j = [s_{j-1}, x_1], \text{ for } 3 \le j \le c, \\ & x_i^{p^{e_i}} = W_i(s_2, \ldots, s_c), \text{ for } 1 \le i \le n \rangle, \end{aligned}$$

and the relator words $W_i(s_2, \ldots, s_c)$ admit the following characterization of mainline vertices. There exists an upper bound $b = b(\mathcal{T}) \ge 2$ such that $\mathfrak{M}$ is a mainline vertex (of depth $\mathrm{dp}(\mathfrak{M}) = 0$) if all words $W_i = W_i(s_2, \ldots, s_b)$ are independent of $s_{b+1}, \ldots, s_c$, for $1 \le i \le n$, and $\mathfrak{M}$ is a vertex of depth $\mathrm{dp}(\mathfrak{M}) \ge 1$ if at least one word $W_{i_0} = W_{i_0}(s_2, \ldots, s_b, s_c)$ with $1 \le i_0 \le n$ depends on s_c, generator of last non-trivial term $\gamma_c \mathfrak{M}$ of lower central series of $\mathfrak{M}$.

1.2.3. *Contribution to the advancement of the scientific field.* For working towards a pro-p presentation $1 \to \mathfrak{R} \to \mathfrak{F} \to G \to 1$ of the p-tower group G, which is the standard form of the complete information contents of the p-class tower $F_p^\infty K$ of a number field K, the amount of information about the p-class tower must be extended in several steps. In this process, we can distinguish two distinct techniques. On the one hand, finding constraints for G which are enforced by an entire class of closely related base fields K, for instance, the existence of an automorphism $\sigma \in \mathrm{Aut}(G)$ induced by the action of $\mathrm{Gal}(K/\mathbb{Q})$, when K is normal [33, § 10, pp. 29–33], or the Shafarevich bounds $\varrho_p \le d_2G \le \varrho_p + r + \theta$ for the relation rank d_2G in the case of fields K sharing a common p-class rank $\varrho_p = d_1G$, signature (r_1, r_2), torsionfree Dirichlet unit rank $r = r_1 + r_2 - 1$, and invariant θ [32, § 1.3, Thm. 1.3], [33, § 3, p. 7], [23, § 5, Thm. 5.1, pp. 28–29]. On the other hand, gathering information structures which are prescribed for G by a single concrete number field K in form of its (non-abelian) *total Artin pattern* $\mathrm{AP}_{\mathrm{tot}}(K)$. The **Successive Approximation Theorem** and, just beyond the limit of its validity, the **Stage Separation Conjecture** admit the step by step identification of higher p-class groups $\mathrm{G}_p^m K$, $m \ge 2$, by means of *iterated* IPADs *of increasing order*, defined recursively by $\tau^{(0)}K := \tau_0 K$, $\tau^{(m)}K := [\tau_0 K; (\tau^{(m-1)}L)_{L \in \mathrm{Lyr}_1 K}]$, $m \ge 1$ [33, § 5, pp. 11-14].
Quadratic fields $\mathbb{Q}(\sqrt{d})$ *of type* $(3,3)$. Until very recently, the length $\ell := \ell_3 K$ of the Hilbert 3-class field tower $K < \mathrm{F}_3^1 K < \mathrm{F}_3^2 K < \ldots < \mathrm{F}_3^\ell K = \mathrm{F}_3^{\ell+1} K = \ldots = \mathrm{F}_3^\infty K$ over a quadratic field $K = \mathbb{Q}(\sqrt{d})$ with 3-class rank $\varrho_3 = 2$, that is, with 3-class group $\mathrm{Cl}_3 K$ of type $(3^u, 3^v)$, $u \ge v \ge 1$, was an open problem [13, § 1.1.2, Prb. 1.1]. Apart from the proven impossibility of an abelian tower with $\ell = 1$ [9, Thm. 4.1.(1)], it was unknown which values $\ell \ge 2$ can occur and whether $\ell = \infty$ is possible or not. In contrast, it is known that $\ell = 1$ for any number field K with 3-class rank $\varrho_3 = 1$, i.e., with non-trivial cyclic 3-class group $\mathrm{Cl}_3 K$, and that $\ell = \infty$ for an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with $d < 0$ and 3-class rank $\varrho_3 \ge 3$ [8]. So the uncertainty was located exactly at $\varrho_3 = 2$.
Solution. The concepts and techniques of the project P 26008-N25 enabled the complete solution of this problem [34, § 3, Thm. 3.1].
Bicyclic biquadratic Eisenstein fields $\mathbb{Q}(\sqrt{-3}, \sqrt{d})$ *of type* $(3,3)$. Except for the unique constant type A.1, $\varkappa(K) = (1,1,1,1)$, all other 22 possible 3-capitulation types [10, Tbl. 6–7, p. 492–493] were realized by quadratic fields K with 3-class group $\mathrm{Cl}_3 K$ of type $(3,3)$, according to [34, § 3, Thm. 3.1]. There arose the question whether fields of higher degree also reveal such a rich variety of different structures. Together with my coworkers, Azizi, Talbi, Talbi, Derhem, we selected complex bicyclic biquadratic fields K with $\mathrm{Cl}_3 K \simeq (3,3)$ as our objects of investigation. It turned out that, while Gauss-Dirichlet-Hilbert fields $K = \mathbb{Q}(\sqrt{-1}, \sqrt{d})$ show a slightly restricted but still comparable variety of types $\varkappa(K)$ [5], Eisenstein-Scholz-Reichardt fields $K = \mathbb{Q}(\sqrt{-3}, \sqrt{d})$ surprise us with a completely opposite behavior, due to an *arithmetical bipolarization* with two distinguished unramified cyclic cubic extensions L_1, L_2 and two isomorphic extensions $L_3 \simeq L_4$ [4]. We succeeded in completely determining all

possibilities for isomorphism types of the second 3-class group $\mathfrak{M} = \mathrm{G}_3^2K$ and proving [13, § 1.4.3, Cnj.1.4]. Computational results underpinning the theory in [4] have been published as integer sequences in the OEIS [14, Project Publications (d.2.2)]. In particular, the sequences A250240 and A250241 together are probably the most extensive construction of a series of 59 Hilbert 3-class fields F_3^1K with absolute degree 36 for which the structure of $\mathrm{Cl}_3\mathrm{F}_3^1K$ was determined. It became feasible by my new funded infrastructure (§ 1.3.1).

Bicyclic biquadratic Dirichlet fields $\mathbb{Q}(\sqrt{-1},\sqrt{d})$ *of type* $(2,2,2)$. Together with my coworkers Azizi, Zekhnini, and Taous [6], we investigated complex bicyclic biquadratic fields $K = \mathbb{Q}(\sqrt{-1},\sqrt{d})$ with 2-class group Cl_2K of type $(2,2,2)$ and radicand $d = p_1p_2q$, composed of primes $p_1 \equiv p_2 \equiv 5 \pmod 8$ and $q \equiv 3 \pmod 4$. As predicted in [11, § 4.2, pp. 451–452], the particular feature of the lattice of intermediate fields between a base field K with $\mathrm{Cl}_2K \simeq (2,2,2)$ and its Hilbert 2-class field F_2^1K is the constitution by *two layers of unramified abelian extensions*, and it seems to be the first time that complete results were given by us for the second layer. One of the seven unramified quadratic extensions of K, denoted by $L_3 := K(\sqrt{q})$, turned out to encapsulate crucial information about the second 2-class group $\mathfrak{M} := \mathrm{Gal}(\mathrm{F}_2^2K/K)$, since $|\mathfrak{M}| = 2 \cdot |\mathrm{Cl}_2L_3|$, and the nilpotency class $c := \mathrm{cl}(\mathfrak{M})$ and the coclass $r := \mathrm{cc}(\mathfrak{M})$ are determined by the abelian type invariants $\mathrm{Cl}_2L_3 \simeq (2^c, 2^{r-1})$. Explicit generators of $\mathfrak{M}$ were given in form of Artin symbols. The paper [6] prepared the solution of [13, § 1.4.4, Cnj 1.5] which is just being finished.

Correction of the Shafarevich Theorem. The main theorem in the paper by Shafarevich [39, Thm. 6, (18′)], which is most important for my entire project, contradicted our joint results in [4], resp. [6], on biquadratic fields containing third, resp. fourth, roots of unity (expressed by the invariant $\theta = 1$). It caused a lot of confusion for several months until I found a fatal misprint in both, the russian original and the english translation. The correction is proved in [23, § 5, Thm. 5.1, pp. 28–29], [4], [29, § 5, Thm. 5.1] [32, § 1.3, Thm. 1.3].

1.2.4. *Breaking of new scientific ground.* **Tree topologies** give a succinct description of the *mutual tree positions* of the second 3-class group $\mathfrak{M} := \mathrm{G}_3^2K$ and the 3-tower group $G := \mathrm{G}_3^\infty K$ of a field K [32, § 5]. Families of *fork* topologies, parametrized with the order $n \ge 0$ of the nth excited state $\uparrow^n$ of a section of IPODs (first layer TKTs) $\varkappa_1K$ for quadratic fields $K = \mathbb{Q}(\sqrt{d})$, where n is defined by the nilpotency class $c := \mathrm{cl}(\mathfrak{M})$ of the second 3-class group $\mathfrak{M}$ of K, have been found by means of trunks with periodic bifurcations and explicit finite Shafarevich covers [17, § 21.2, pp. 184–193], [20, § 7, pp. 751–756], for the smallest non-trivial coclass $r := \mathrm{cc}(\mathfrak{M}) = 2$. Several scenarios are shown in [33, § 7, p. 22]. The way to the symmetric fork in the simple type scenario is explained in broadest detail in [33, § 7, Thm., pp. 16–21], which is a high level update of my story with title “11. Pattern recognition / 11.1. Historical example” in my Wikipedia article [14, Project publications, (d.1.2)]. Theorem E in [33, § 7, p. 22] together with infinite pro-3 groups in [15, § 3.6, Thm 3.6] proves [13, § 1.4.1, Cnj. 1.1]. In contrast to these proven tree topologies for $r = 2$, we are still working on much more complicated tree topologies for $r \in \{4,6\}$ with my coauthor M. F. Newman [36]. The increase of complexity is due firstly to *multifurcations* with higher

step sizes $s \in \{3,4\}$ instead of the bifurcations with $s = 2$, and secondly to the occurrence of *two scaffold types* b and d instead of the single scaffold type c. In this situation, which will be a main topic of [35], tree topologies encapsulate invaluable key information about secret navigation paths through very dense descendant trees, for instance:
Simple sporadic type F for $d < 0$: $r = 2m + 4$, $c = r + 1$, and the topology symbol is

$$\overbrace{\mathrm{F}\binom{2}{\rightarrow}}^{\text{Leaf}}\ \overbrace{\left\{\mathrm{b}\binom{2}{\rightarrow}\right\}^{2m}}^{\text{Scaffold 1}}\ \overbrace{\mathrm{b}}^{\text{Fork}}\ \overbrace{\left\{\binom{4}{\leftarrow}\mathrm{b}\binom{2}{\leftarrow}\mathrm{b}\right\}^{m}}^{\text{Scaffold 1}}\ \overbrace{\binom{4}{\leftarrow}\mathrm{F}\binom{2}{\leftarrow}\mathrm{F}\binom{4}{\leftarrow}\mathrm{F}}^{\text{Link}}\ \overbrace{\left\{\binom{1}{\leftarrow}\mathrm{F}\binom{2}{\leftarrow}\mathrm{F}\right\}^{m+1}}^{\text{Path}}$$

Simple periodic type F for $d < 0$: $r = 2m + 4$, $c = r + 2n + 3$, and the topology symbol is

$$\overbrace{\mathrm{F}\binom{1}{\rightarrow}}^{\text{Leaf}}\ \overbrace{\left\{\mathrm{d}\binom{1}{\rightarrow}\right\}^{2n+1}}^{\text{Mainline}}\ \overbrace{\left\{\mathrm{d}\binom{2}{\rightarrow}\right\}}^{\text{Scaf. 2}}\ \overbrace{\left\{\mathrm{b}\binom{2}{\rightarrow}\right\}^{2m}}^{\text{Scaffold 1}}\ \overbrace{\mathrm{b}}^{\text{Fork}}\ \overbrace{\left\{\binom{4}{\leftarrow}\mathrm{b}\binom{2}{\leftarrow}\mathrm{b}\right\}^{m}}^{\text{Scaffold 1}}\ \overbrace{\left\{\binom{4}{\leftarrow}\mathrm{d}\binom{2}{\leftarrow}\mathrm{d}\right\}^{n+1}}^{\text{Scaffold 2}}\ \overbrace{\binom{4}{\leftarrow}\mathrm{F}\binom{2}{\leftarrow}\mathrm{F}\binom{4}{\leftarrow}\mathrm{F}}^{\text{Link}}\ \overbrace{\left\{\binom{1}{\leftarrow}\mathrm{F}\binom{2}{\leftarrow}\mathrm{F}\right\}^{n+2}}^{\text{Path}}$$

The occurrence of the conspicuous link with path length 3 in both cases is still a mystery.

1.3. Information on the use of available funds.

1.3.1. *Major items of equipment purchased.* For the reasons explained in detail in [19], my application for funding of project-specific equipment was granted by the FWF. I got 9016.60 Euros for purchasing a workstation (Lenovo ThinkStation D30 with two Intel XEON 8-core processors and 256 GB random access memory) and an uninterruptible power supply (APC Smart UPS 1kW).

2. Personnel development. Joint work with number theorists in Morocco resulted in a series of *five invited course lectures* within the frame of the UNESCO supported Research School CIMPA Oujda in May 18–29, 2015, and a *crowning completion* of the FWF project P 26008-N25 by an *invited keynote* at the Symposium ANCI Taza in November 11–12, 2016, [33]. Intensive publishing activity with SCIRP in Wuhan, China, was rewarded by *invited lectures* at the ICGA Shanghai in July 2015 [21], and the ICGA Suzhou in July 2016, [31]. Due to an increasing number of publications and to considerable public interest in downloading my papers, *my impact factor* in ResearchGate improved to a remarkable value of 16.35. Furthermore, I received the honour of working as a *referee* for the Int. J. Number Theory, the Asian-European J. Math., and the J. Algebra Appl, and the invitation to write a chapter *Modeling rooted in-trees by finite p-groups* for the open access book *Graph Theory*, edited by B. Sirmacek, InTech, [35].

3. Effects beyond the scientific field. Inspired by *joint projects with artists* and *discussions with theorists* at the University of Arts in Graz, I used my expertise concerning non-abelian groups, which I acquired within the frame of the project P 26008-N25, and my long experience as a concert pianist, for writing a *mathematically rigorous article* on neo-Riemannian harmony theory [26], underpinned by an illustrating composition in the *style of romantic expressionism* [27].

4. International conferences, lectures, courses, and symposia.

(1) D. C. Mayer, *Finite* 3*-groups as viewed from class field theory*, with coauthor M. F. Newman, Groups St. Andrews 2013, University of St. Andrews, Fife, Scotland, UK, contributed presentation, August 11, 2013.

(2) D. C. Mayer, 3*-class field towers of exact length* 3, with coauthors M. R. Bush and M. F. Newman, 18. Congress of the Austrian Mathematical Society and 123. Annual Meeting of the German Mathematical Union 2013, University of Innsbruck, Tyrol, Austria, contributed presentation, September 24, 2013.

(3) D. C. Mayer, *Class towers and capitulation over quadratic fields*, with coauthors M. R. Bush and M. F. Newman, West Coast Number Theory 2013, Asilomar Conference Center, Pacific Grove, Monterey, California, USA, contributed presentation, December 18, 2013.

(4) D. C. Mayer, *La Théorie Algorithmique des Nombres, Construction des Corps de Nombres Algébriques et Corps de Classes*, École de Recherche CIMPA (Centre International de Mathématiques Pures et Appliquées) UNESCO Maroc 2015, Théorie des Nombres et ses Applications, Université Mohammed Premier, Faculté des Sciences d'Oujda, Morocco, five invited course lectures, May 18–29, 2015.

(5) D. C. Mayer, *Index-p abelianization data of p-class tower groups*, 29ièmes Journées Arithmétiques 2015, Univ. of Debrecen, Hungary, contributed presentation, July 09, 2015.

(6) D. C. Mayer, *Periodic sequences of p-class tower groups*, 1st International Conference on Groups and Algebras 2015, Shanghai, China, invited lecture, July 21, 2015.

(7) D. C. Mayer, *New number fields with known p-class tower*, 22nd Czech and Slovak International Conference on Number Theory 2015, Liptovský Ján, Slovakia, contributed presentation, August 31, 2015.

(8) D. C. Mayer, *p-Capitulation over number fields with p-class rank two*, 2nd International Conference on Groups and Algebras 2016, Suzhou, China, invited lecture, July 26, 2016.

(9) D. C. Mayer, *Recent progress in determining p-class field towers*, 1st International Colloquium of Algebra, Number Theory, Cryptography and Information Security (ANCI) 2016, Faculté Polydisciplinaire de Taza, Université Sidi Mohamed Ben Abdellah, Morocco, invited keynote, November 12, 2016.

References

[1] M. Arrigoni, *On Schur σ-groups*, Math. Nachr. **192** (1998), 71–89.

[2] E. Artin, *Beweis des allgemeinen Reziprozitätsgesetzes*, Abh. Math. Sem. Univ. Hamburg **5** (1927), 353–363, DOI 10.1007/BF02952531.

[3] E. Artin, *Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz*, Abh. Math. Sem. Univ. Hamburg **7** (1929), 46–51, DOI 10.1007/BF02941159.

[4] A. Azizi, M. Talbi, Mm. Talbi, A. Derhem, D. C. Mayer, *The group* $\mathrm{Gal}(k_3^{(2)}|k)$ *for* $k = \mathbb{Q}(\sqrt{-3}, \sqrt{d})$ *of type* $(3, 3)$, Int. J. Number Theory (2016), DOI 10.1142/S1793042116501207.

[5] A. Azizi, M. Talbi, Mm. Talbi, D. C. Mayer, *The isomorphism type of the Galois group* $\mathrm{Gal}(k_3^{(2)}|k)$ *for* $k = \mathbb{Q}(\sqrt{-1}, \sqrt{d})$ *of type* $(3, 3)$, in preparation.

[6] A. Azizi, A. Zekhnini, M. Taous, D. C. Mayer, *Principalization of* 2*-class groups of type* $(2, 2, 2)$ *of biquadratic fields* $\mathbb{Q}(\sqrt{p_1p_2q}, \sqrt{-1})$, Int. J. Number Theory **11** (2015), no. 4, 1177 – 1216, DOI 10.1142/S1793042115500645.

[7] M. R. Bush and D. C. Mayer, 3*-class field towers of exact length* 3, J. Number Theory **147** (2015), 766–777, DOI 10.1016/j.jnt.2014.08.010.

[8] H. Koch und B. B. Venkov, *Über den p-Klassenkörperturm eines imaginär-quadratischen Zahlkörpers*, Astérisque **24–25** (1975), 57–67.

[9] D. C. Mayer, *The second p-class group of a number field*, Int. J. Number Theory **8** (2012), no. 2, 471–505, DOI 10.1142/S179304211250025X.

[10] D. C. Mayer, *Transfers of metabelian p-groups*, Monatsh. Math. **166** (2012), no. 3–4, 467–495, DOI 10.1007/s00605-010-0277-x.

[11] D. C. Mayer, *The distribution of second p-class groups on coclass graphs*, J. Théor. Nombres Bordeaux **25** (2013), no. 2, 401–456, DOI 10.5802/jtnb.842.

[12] D. C. Mayer, *Principalization algorithm via class group structure*, J. Théor. Nombres Bordeaux **26** (2014), no.2, 415–464, DOI 10.5802/jtnb.874.

[13] D. C. Mayer, Proposal for a stand-alone project, *Towers of p-class fields over algebraic number fields*, 2013, `http://www.algebra.at/Proposal20130213.pdf`.

[14] D. C. Mayer, *Website for Principal Investigators of the FWF*, 2013, `http://www.algebra.at/FWFP26008N25/index.html`.

[15] D. C. Mayer and M. F. Newman, *Finite* 3*-groups as viewed from class field theory*, Groups St. Andrews 2013, Univ. of St. Andrews, Fife, Scotland, UK, contributed presentation delivered on 11 August 2013. `http://www.algebra.at/GroupsStAndrews2013.pdf`.

[16] D. C. Mayer, *Two-stage towers of p-class fields for $p \geq 5$*, Preprint, 2014, `https://www.researchgate.net/publication/267099298_Two-stage_towers_of_p-class_fields_for_p5`.

[17] D. C. Mayer, *Periodic bifurcations in descendant trees of finite p-groups*, Adv. Pure Math. **5** (2015), no. 4, 162–195, DOI 10.4236/apm.2015.54020, Special Issue on Group Theory, March 2015.

[18] D. C. Mayer, *Index-p abelianization data of p-class tower groups*, Adv. Pure Math. **5** (2015) no. 5, 286–313, DOI 10.4236/apm.2015.55029, Special Issue on Number Theory and Cryptography, April 2015.

[19] D. C. Mayer, *Application for funding of project-specific equipment*, 2015, `http://www.algebra.at/ApplicationEquipment.pdf`.

[20] D. C. Mayer, *Periodic sequences of p-class tower groups*, J. Appl. Math. Phys. **3** (2015), no. 7, 746–756, DOI 10.4236/jamp.2015.37090.

[21] D. C. Mayer, *Periodic sequences of p-class tower groups*, 1st International Conference on Groups and Algebras 2015, Shanghai, China, invited lecture delivered on July 21, 2015, `http://www.algebra.at/PresICGA2015.pdf`.

[22] D. C. Mayer, *New number fields with known p-class tower*, 22nd Czech and Slovak International Conference on Number Theory 2015, Liptovský Ján, Slovakia, presentation delivered on August 31, 2015, `http://www.algebra.at/22CSICNT2015.pdf`.

[23] D. C. Mayer, *New number fields with known p-class tower*, Tatra Mt. Math. Pub., **64** (2015), 21–57, DOI 10.1515/tmmp-2015-0040, Special Issue on Number Theory and Cryptology ‘15.

[24] D. C. Mayer, *Artin transfer patterns on descendant trees of finite p-groups*, Adv. Pure Math. **6** (2016), no. 2, 66 – 104, DOI 10.4236/apm.2016.62008, Special Issue on Group Theory Research, January 2016.

[25] D. C. Mayer, *Index-p abelianization data of p-class tower groups, II.* (arXiv: 1601.00179v1 [math.NT] 02 Jan 2016.)

[26] D. C. Mayer, *Uniform triadic transformations as viewed from group theory*, 2015, `http://www.algebra.at/UTTandGroups.pdf`.

[27] D. C. Mayer, *Prélude en la majeur*, `http://www.algebra.at/PreludeA.pdf`.

[28] D. C. Mayer, *Annihilator ideals of two-generated metabelian p-groups.* (arXiv: 1603.09288v1 [math.GR] 30 Mar 2016.)

[29] D. C. Mayer, *Three-stage towers of* 5*-class fields.* (arXiv: 1604.06930v1 [math.NT] 23 Apr 2016.)

[30] D. C. Mayer, *p-Capitulation over number fields with p-class rank two*, J. Appl. Math. Phys. **4** (2016), no. 7, 1280–1293, DOI 10.4236/jamp.2016.47135.

[31] D. C. Mayer, *p-Capitulation over number fields with p-class rank two*, 2nd International Conference on Groups and Algebras 2016, Suzhou, China, invited lecture delivered on July 26, 2016, `http://www.algebra.at/ICGA2016Suzhou.pdf`.

[32] D. C. Mayer, *Recent progress in determining p-class field towers*, Gulf J. Math. (Dubai, UAE). (arXiv: 1605.09617v1 [math.NT] 31 May 2016.)

[33] D. C. Mayer, *Recent progress in determining p-class field towers*, 1st International Colloquium of Algebra, Number Theory, Cryptography and Information Security 2016, Faculté Polydisciplinaire de Taza, Université Sidi Mohamed Ben Abdellah, Morocco, invited keynote delivered on November 12, 2016, `http://www.algebra.at/ANCI2016DCM.pdf`.

[34] D. C. Mayer, *Criteria for three-stage towers of p-class fields.* (arXiv: 1601.00179v2 [math.NT] 28 Nov 2016.)

[35] D. C. Mayer, *Modeling rooted in-trees by finite p-groups*, to appear in the book *Graph Theory*, Ed. B. Sirmacek, InTech, 2017, `http://www.algebra.at/GraphTheory.pdf`.

[36] D. C. Mayer and M. F. Newman, *Finite* 3*-groups with transfer kernel type* F, in preparation, `http://www.algebra.at/TransferSectionF.pdf`.

[37] K. Miyake, *Algebraic investigations of Hilbert's Theorem* 94*, the principal ideal theorem and the capitulation problem*, Expo. Math. **7** (1989), 289–346.

[38] K. Miyake, *Some p-groups with two generators which satisfy certain conditions arising from arithmetic in imaginary quadratic fields*, Tôhoku Math. J. **44** (1992), 443–469.

[39] I. R. Shafarevich, *Extensions with prescribed ramification points* (Russian), Publ. Math., Inst. Hautes Études Sci. **18** (1964), 71–95. (English transl. by J. W. S. Cassels in Amer. Math. Soc. Transl., II. Ser., **59** (1966), 128–149.)

[40] H. Suzuki, *A generalization of Hilbert's Theorem* 94, Nagoya Math. J. **121** (1991), 161–169.

Naglergasse 53, 8010 Graz, Austria
E-mail address: `quantum.algebra@algebra.at`
URL: `http://www.algebra.at`