UNIFORM TRIADIC TRANSFORMATIONS AS VIEWED FROM GROUP THEORY

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Abstract. We show that finite non-abelian groups, more precisely, semidirect products and wreath products, with interesting algebraic properties can be used for a sound foundation of the musical transformation theories developed by Riemann, Lewin, Hyer, Cohn, and Hook.

1. Introduction

The neo-Riemannian operations [3] and the uniform triadic transformations [5] provide a way of describing the harmonic time evolution of music. Both are based on the concept of *triads*, which we recall in section § 2 about pitch classes and their transposition and inversion. In section § 3, we investigate uniform triadic transformations (UTTs), and section § 4 is devoted to the study of neo-Riemannian operations (NROs). This article was inspired by the preprint [12] of Ada Zhang, who got her B.A. in music from the University of Washington at Seattle, and is currently a piano teacher at Pittsburgh, PA.

2. Pitch classes, transpositions and inversion

The *scale degree* only admits a coarse characterization of a triad independently of its tonic and of its mode. The major triad XM and the minor triad xm, with respect to any tonic X, resp. x, are both represented by the set $[\hat{1},\hat{3},\hat{5}]$ of scale degrees. With *pitch class* sets, however, we are able to fix the tonic, for instance C, and to distinguish the major triad CM, $[0,4,7]$, which is a symbol for $[c,e,g]$, from the minor triad cm, $[0,3,7]$, which is a symbol for $[c,e^\flat,g]$. An overview of scale degrees and pitch classes is given in Table 1.

Table 1. Scale degrees and pitch classes

Scale degree	$\hat{1}$		$\hat{2}$		$\hat{3}$	$\hat{4}$		$\hat{5}$		$\hat{6}$		$\hat{7}$
Sharp		$c^\sharp$					$f^\sharp$					
Tonic	c		d		e	f		g		a		b
Flat		$d^\flat$		$e^\flat$			$g^\flat$		$a^\flat$		$b^\flat$	
Pitch class	0	1	2	3	4	5	6	7	8	9	10	11

Proposition 2.1. *The pitch classes, taken with respect to the equally tempered scale, together with the operation of addition modulo* 12*, form a group* $P=\{0,1,\ldots,11\}$ *isomorphic to the cyclic group* C_{12} *of order* 12*, which has the identifier* $\langle 12,2\rangle$ *in the SmallGroups Database* [1, 2, 4, 9]*. In particular,* P *is an abelian (or commutative) finite group.*

Proof. The neutral element of the group P is 0. A generator of order 12 is the element 1. Thus, $P\simeq C_{12}\simeq\langle 12,2\rangle$ is isomorphic to the cyclic group of order 12. Every cyclic group is abelian. □

Date: February 23, 2016.

2000 *Mathematics Subject Classification.* Primary 00A65, 20E22; secondary 97M80.

Key words and phrases. Sets of pitch classes, uniform triadic transformations, neo-Riemanian operations, semidirect product, wreath product, mathematical modeling.

Research supported by the Austrian Science Fund (FWF): P 26008-N25.

Since we would like to have a means for changing the tonic, we introduce two kinds of permutations of P, that is, bijective mappings from P onto itself. The first kind consists of *transpositions*,

$$T_n : P \to P, \quad p \mapsto T_n(p) \equiv p + n \pmod{12}, \tag{2.1}$$

which move a pitch class up by n semitones, for any fixed non-negative integer n. A generic paradigm of the second kind is the *inversion*,

$$I : P \to P, \quad p \mapsto I(p) \equiv -p \pmod{12}, \tag{2.2}$$

which generates the mirror image of a pitch class with respect to the fixed pitch class 0 as the centre of reflection.

Theorem 2.1. *The transposition T_1 and the inversion I generate a group $\Pi = \langle T_1, I\rangle$ with respect to the (usual right to left) composition of mappings. This non-abelian group Π is isomorphic to the dihedral group $D_{12} = C_{12} \rtimes C_2$ of order 24, that is the semidirect product of two cyclic groups C_{12} and C_2, which has the identifier $\langle 24, 6\rangle$ in the SmallGroups Library* [2]. *The transpositions alone form a normal subgroup $T = \{T_0 = 1, T_1, \ldots, T_{11}\} \lhd \Pi$ isomorphic to the cyclic group $C_{12} \simeq \langle 12, 2\rangle$ of order 12. The inversion generates a malnormal subgroup $\{1, I\} < \Pi$ isomorphic to the cyclic group $C_2 \simeq \langle 2, 1\rangle$ of order 2. A presentation of Π can be given in the shape*

$$\Pi = \langle T_1, I \mid T_1^{12} = 1,\ I^2 = 1,\ T_1 I = I T_1^{-1}\rangle. \tag{2.3}$$

Proof. We point out the following facts.

(1) The smallest power T_1^n of T_1 with positive exponent $n \geq 1$ which coincides with the identity map is $T_1^{12} = 1$, since $T_1^n(p) \equiv p + n \pmod{12}$ for all $p \in P$.
(2) The inversion I is an involution of order 2, since it is its own inverse with $I^2 = 1$.
(3) The inverse T_1^{-1} of T_1 coincides with T_{11} and the commutator relation can be verified by $IT_1^{-1}(p) \equiv IT_{11}(p) \equiv I(p + 11) \equiv -p - 11 \equiv -p + 1 \equiv T_1(-p) \equiv T_1 I(p) \pmod{12}$, which is valid for any $p \in P$.
(4) The presentation $\Pi = \langle T_1, I \mid T_1^{12} = 1,\ I^2 = 1,\ T_1 I = I T_1^{-1}\rangle$ of the group Π shows that $\Pi \simeq D_{12} = C_{12} \rtimes C_2$ is isomorphic to the dihedral group of order 24.
(5) The required *homomorphism* $\theta : C_2 \to \mathrm{Aut}(C_{12})$, which is tacitly underlying the semidirect product $D_{12} = C_{12} \rtimes C_2$, has the trivial property $\theta(1) = 1$ and the essential property

$$\theta(I)(T_1) = I^{-1} T_1 I = I^{-1} I T_1^{-1} = T_1^{-1} (= T_{11}).$$

□

With transpositions alone it would be impossible to switch from a major triad to its relative minor triad. Employing the inversion, however, we can describe such a harmonic evolution:

Example 2.1. The major triad CM, $[0, 4, 7]$, is mapped to

$$T_7 I([0, 4, 7]) = [T_7 I(0), T_7 I(4), T_7 I(7)] = [-0 + 7, -4 + 7, -7 + 7] = [7, 3, 0],$$

by the permutation $T_7 I$, that is, to the minor triad cm, $[0, 3, 7]$.

A drawback is that mapping the major triad DM, $[2, 6, 9]$, to the minor triad dm, $[2, 5, 9]$, requires a different permutation $T_{11} I$ as the following chain of equations shows:

$$T_{11} I([2, 6, 9]) = [T_{11} I(2), T_{11} I(6), T_{11} I(9)] = [-2 + 11, -6 + 11, -9 + 11] = [9, 5, 2],$$

whereas our listening experience causes the expectation of a uniform triadic transformation for both mappings.

Without giving all steps, we emphasize that $T_7 I([2, 6, 9]) = [1, 5, 10]$, the b$^\flat$m triad, and $T_{11} I([0, 4, 7]) = [4, 7, 11]$, the em triad.

This was one of the reasons why Julian Hook developed his theory of *uniform triadic transformations* [5]. Hook received his Ph.D. in mathematics from Princeton University, NJ, and his D.Mus. from Jacobs School of Music, Indiana University Bloomington.

3. Uniform triadic transformations

Hook defines a *triad* to be an ordered pair $\Delta = (r, \sigma)$ consisting of a *root* $r \in \{0, 1, \dots, 11\}$ and a *mode* or *sign* $\sigma \in \{+, -\}$ which is an abbreviation for $\{+1, -1\}$.

For instance, $\Delta = (0, +)$ is the C major triad and $\Delta = (6, -)$ is the f$^\sharp$ minor triad.

Proposition 3.1. *Together with the operation*

$$\Delta_1 \Delta_2 = (r_1, \sigma_1)(r_2, \sigma_2) = (r_1 + r_2, \sigma_1 \sigma_2), \tag{3.1}$$

the triads form an abelian group Γ *of order* 24 *isomorphic to the direct product* $C_{12} \times C_2$ *of cyclic groups. It has the identifier* $\langle 24, 9 \rangle$ *in the SmallGroups Library* [2].

A Γ-*interval* is defined between triads, rather than between pitch classes, to be a pair $\mathrm{int}(\Delta_1, \Delta_2) = (t, \sigma)$ consisting of the *transposition level* $t = r_2 - r_1$ and the *sign factor* $\sigma = \sigma_1 \sigma_2$.

Proposition 3.2. *If a (general) triadic transformation is defined to be a bijective mapping from* Γ *onto itself, that is a permutation* $T : \Gamma \to \Gamma$*, then the group* $\mathcal{G}$ *of all triadic transformations is isomorphic to the symmetric group* S_{24} *of degree* 24 *and order*

$$24! = 24 \cdot 23 \cdots 2 \cdot 1 = 620\,448\,401\,733\,239\,439\,360\,000.$$

To avoid the problems mentioned at the end of the previous section § 2, Hook demands the validity of the following axiom for a *uniform triadic transformation*, $U : \Gamma \to \Gamma$ mapping a pre-image $\Delta = (r, \sigma)$ to an image $\Delta' = (r', \sigma')$.

$$U(r, \sigma) = (r', \sigma') \quad \Longrightarrow \quad U(r + t, \sigma) = (r' + t, \sigma'), \text{ for all } 0 \le t \le 11. \tag{3.2}$$

A uniform triadic transformation (UTT) $U = \langle \sigma, t^+, t^- \rangle$ is completely determined by three invariants, the *transposition level for major triads* t^+, the *transposition level for minor triads* t^-, both elements of $\{0, 1, \dots, 11\}$, and the *sign* $\sigma \in \{+, -\}$.

U is called *mode preserving* if $\sigma = +$ and *mode reversing* if $\sigma = -$.

The *action* of a UTT $U = \langle \sigma, t^+, t^- \rangle$ on a major triad $\Delta = (r, +)$, resp. minor triad $\Delta = (r, -)$, is defined by

$$(r, +)U = (r + t^+, \sigma), \text{ resp. } (r, -)U = (r + t^-, -\sigma). \tag{3.3}$$

Theorem 3.1. *With respect to Hook's left to right composition of mappings,*

$$UV = \langle \sigma_U, t_U^+, t_U^- \rangle \langle \sigma_V, t_V^+, t_V^- \rangle = \langle \sigma_U \sigma_V, t_U^+ + t_V^{\sigma_U}, t_U^- + t_V^{-\sigma_U} \rangle, \tag{3.4}$$

the set $\mathcal{U}$ *of all uniform triadic transformations* $U : \Gamma \to \Gamma$ *becomes a non-abelian group of order* $12^2 \cdot 2 = 144 \cdot 2 = 288$*, isomorphic to the wreath product* $C_{12} \wr S_2$ *of* C_{12} *with* S_2*, that is the semidirect product* $(C_{12} \times C_{12}) \rtimes S_2$ *of* $C_{12} \times C_{12}$ *with* S_2*.*

Remark 3.1. With respect to the usual right to left composition of mappings,

$$V \circ U = \langle \sigma_V, t_V^+, t_V^- \rangle \circ \langle \sigma_U, t_U^+, t_U^- \rangle = UV = \langle \sigma_U, t_U^+, t_U^- \rangle \langle \sigma_V, t_V^+, t_V^- \rangle = \langle \sigma_V \sigma_U, t_V^{\sigma_U} + t_U^+, t_V^{-\sigma_U} + t_U^- \rangle,$$

the group operation on $\mathcal{U}$ coincides with the usual group operation,

$$V \circ U = \langle t_V^+, t_V^-, \sigma_V \rangle \circ \langle t_U^+, t_U^-, \sigma_U \rangle = \langle \theta(\sigma_U)(t_V^+) + t_U^+, \theta(\sigma_U)(t_V^-) + t_U^-, \sigma_V \sigma_U \rangle,$$

on the wreath product $C_{12} \wr S_2$, where the *homomorphism* $\theta : S_2 \to \mathrm{Aut}(C_{12} \times C_{12})$, is given by the identity $\theta(+) = 1$ and the twisting automorphism $\theta(-)(t_1, t_2) = (\theta(-)(t_1), \theta(-)(t_2)) = (t_2, t_1)$.

Corollary 3.1. *The inverse element* U^{-1} *of a uniform triadic transformation* $U = \langle \sigma, t^+, t^- \rangle$ *is given by*

$$U^{-1} = (\sigma, -t^{\sigma}, -t^{-\sigma}) \tag{3.5}$$

4. Neo-Riemannian operations

David Lewin investigated contextual inversions of triads, in particular the parallel operation PAR and the relative operation REL, in his 1982 article *A formal theory of generalized tonal functions* [7]. In 1987, he extended his theory in *Generalized musical intervals and transformations* (GMIT) [8] with Riemann's leading tone exchange LTE and two kinds of shift operations, the submediant shift MED and the subdominant shift DOM. Brian Hyer developed Lewin's theory further in his 1989 dissertation [6].

Definition 4.1. The three *neo-Riemannian operations* can be expressed with the aid of uniform triadic transformations: the *parallel operation* (PAR) is the UTT $P = \langle -, 0, 0\rangle$, the *leading tone exchange* (LTE) is the UTT $L = \langle -, 4, 8\rangle$, and the *relative operation* (REL) is the UTT $R = \langle -, 9, 3\rangle$ [11].

P preserves the perfect fifth of a consonant triad,
L preserves the minor third, and
R preserves the major third.

Remark 4.1. According to our conviction and feeling, the concepts *parallel* and *relative* should be defined just the other way round. (There is no reason to call CM and cm parallel.)

The neo-Riemannian operations give rise to one of the most beautiful geometric interpretations of the *orbits* under the action of the group $\mathcal{U}$ of uniform triadic transformations on the set Γ of triads as subsets of a differential 3-manifold in Euclidean 4-space: the *hyper torus* $\mathbb{T}^3$ [3].

Theorem 4.1. *The neo-Riemannian subgroup* $\mathcal{R} = \langle P, L, R\rangle$ *of* $\mathcal{U}$ *is generated by the mode reversing involutions* P, L, R *of order* 2 *and consists of all UTTs of the shape* $\langle \sigma, t_1, t_2\rangle$ *with* $t_1 + t_2 \equiv 0 \pmod{12}$ *and arbitrary sign* σ. *It is isomorphic to the dihedral group* $D_{12} = C_{12} \rtimes C_2$ *of order* 24*, that is the semidirect product of two cyclic groups* C_{12} *and* C_2*, which has the identifier* $\langle 24, 6\rangle$ *in the SmallGroups Database* [2].

The mode preserving transformations in $\mathcal{R}$ *form an abelian subgroup* $\mathcal{S}$ *of index* $(\mathcal{R} : \mathcal{S}) = 2$ *which contains the three composita*
dominant shift (DOM) $D = R \circ L = LR$ *of order* 12*,*
$D^4 = P \circ L = LP$ *of order* 3*, and*
$D^3 = P \circ R = RP$ *of order* 4*.*
The orders $12, 3, 4$ *of these three transformations give rise to the corresponding period lengths* $12, 3, 4$ *of lattice points on the three* 1*-spheres (i.e. circles) whose topological product constitutes the* 3*-torus* $\mathbb{T}^3$*.*

The four double orbits of $D^4 = P \circ L = LP$ *in* Γ *are exactly the four hexatonic systems of Cohn* [3] *which together form the hyper-hexatonic system of Robert Cook and the hyper-toroidal Tonnetz of Hyer:*
first double orbit: {CM, EM, A$^\flat$M; cm, a$^\flat$m, em}*;*
second double orbit: {GM, BM, E$^\flat$M; gm, e$^\flat$m, bm}*;*
third double orbit: {DM, F$^\sharp$M, B$^\flat$M; dm, b$^\flat$m, f$^\sharp$m}*;*
fourth double orbit: {AM, C$^\sharp$M, FM; am, fm, c$^\sharp$m}.

Proof. From $P = \langle -, 0, 0\rangle$, $L = \langle -, 4, 8\rangle$, $R = \langle -, 9, 3\rangle$ it follows that
$R \circ L = LR = \langle -, 4, 8\rangle\langle -, 9, 3\rangle = \langle +, 4+3, 8+9\rangle = \langle +, 7, 5\rangle$,
$P \circ L = LP = \langle -, 4, 8\rangle\langle -, 0, 0\rangle = \langle +, 4, 8\rangle$,
$P \circ R = RP = \langle -, 9, 3\rangle\langle -, 0, 0\rangle = \langle +, 9, 3\rangle$.
Computing successive powers yields
$(P \circ R)^2 = \langle +, 6, 6\rangle$, $(P \circ R)^3 = \langle +, 3, 9\rangle$, $(P \circ R)^4 = \langle +, 0, 0\rangle = 1$;
$(P \circ L)^2 = \langle +, 8, 4\rangle$, $(P \circ L)^3 = \langle +, 0, 0\rangle = 1$;
$(R \circ L)^2 = \langle +, 2, 10\rangle$, $(R \circ L)^3 = \langle +, 9, 3\rangle = P \circ R$, $(R \circ L)^4 = \langle +, 4, 8\rangle = P \circ L$.
Consequently, $(R \circ L)^{12} = (P \circ R)^4 = (P \circ L)^3 = 1$,
whence $\mathrm{ord}(P \circ R) = 4$, $\mathrm{ord}(P \circ L) = 3$, and $\mathrm{ord}(R \circ L) = 12$.

Finally, we show the isomorphism of $\mathcal{R}$ to the dihedral group D_{12}.
It suffices to consider the conjugate of $D = R \circ L$ with L:

$L^{-1}DL = LDL = \langle -, 4, 8\rangle\langle +, 7, 5\rangle\langle -, 4, 8\rangle = \langle -, 9, 3\rangle\langle -, 4, 8\rangle = \langle +, 5, 7\rangle = D^{-1}$.
Thus, a presentation of the neo-Riemannian subgroup $\mathcal{R}$ can be given in the shape

$$\mathcal{R} = \langle D, L \mid D^{12} = 1,\ L^2 = 1,\ DL = LD^{-1}\rangle. \tag{4.1}$$

□

We point out that the transposition levels 7, 5 of the order 12 transformation $R \circ L$ are coprime to 12, whereas those of the order 4 transformation $P \circ R$, i.e. 9, 3, have the greatest common divisor (gcd) 3 with 12, and those of the order 3 transformation $P \circ L$, i.e. 4, 8, have the gcd 4 with 12.

5. A detailed example

Example 5.1. Table 2 associates a unique triad, resp. two triads, with each bar of the Prelude in A major shown in the Appendix. An additional flavor indicates an extension of the triad to a chord consisting of four pitch classes.

Table 2. Harmonic evolution of the Prelude in the Appendix

Bars	0	1, 2	3, 4	5, 6	7, 8	9	10	11
Triads	(4, +)	(9, +)	(4, +)	(9, +)	(4, +)	(9, +), (6, −)	(6, −)	(6, −)
	E	A	E	A	E	$A, f^\sharp$	$f^\sharp$	$f^\sharp$
Flavor			7−		7−			6+
Bars	12	13	14	15	16, 17	18	19	20
Triads	(1, +)	(6, −)	(1, +)	(6, −)	(1, +), (11, −)	(1, +), (11, −)	(11, −)	(4, +)
	$C^\sharp$	$f^\sharp$	$C^\sharp$	$f^\sharp$	$C^\sharp, b$	$C^\sharp, b$	b	E
Flavor	7−		7−		6+	6+	6+	7−

We can express the harmonic evolution by neo-Riemannian operations in an invariant way, i.e., independently of transpositions:
$D = \langle +, 7, 5\rangle$ (DOM) from bar 2 to 3, resp. bar 6 to 7,
$D^{-1} = \langle +, 5, 7\rangle$ from bar 0 to 1, resp. bar 4 to 5, resp. bar 8 to 9,
$R = \langle -, 9, 3\rangle$ (REL) within bar 9,
$PD = \langle -, 5, 7\rangle$ (PAR DOM) from bar 11 to 12, resp. bar 13 to 14,
$(PD)^{-1} = PD$, which is its own inverse, from bar 12 to 13, resp. bar 14 to 15,
$D = \langle +, 7, 5\rangle$ (DOM) from bar 15 to 16,
and finally $D^{-1}RD = \langle -, 7, 5\rangle$ from bar 19 to 20.
The big advantage of neo-Riemannian operations is the independence of transpositions, for instance a transposition by a full tone up, from A major to B major. Then we have the following changes of triads $A \to B$, $E \to F^\sharp$, $f^\sharp \to a^\flat$, $C^\sharp \to E^\flat$ and $b \to d^\flat$, but the NROs remain the same.

6. Acknowledgements

The author would like to express his gratitude to Erica Bisesi from the Centre of Systematic Music Theory at the University of Arts in Graz, Austria, for stimulating our interest in studying the connections between musical expressivity and mathematical structures.

We gratefully acknowledge that our research is supported by the Austrian Science Fund (FWF): P 26008-N25.

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