Principalization Algorithm
via Class Group Structure

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16:15 – 16:45
Author: Daniel C. Mayer
Abstract.

For an algebraic number field $K$ with 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$ or $(9, 3)$, the structure of the 3-class groups $\text{Cl}_3(N_i)$ of the four unramified cyclic cubic extension fields $N_i$, $1 \leq i \leq 4$, of $K$ is calculated with the aid of presentations for the metabelian Galois group $G_3^2(K) = \text{Gal}(F_3^2(K)/K)$ of the second Hilbert 3-class field $F_3^2(K)$ of $K$. In the case of a quadratic base field $K = \mathbb{Q}(\sqrt{D})$ it is shown that the structure of the 3-class groups of the four $S_3$-fields $N_1, \ldots, N_4$ determines the type of principalization of the 3-class group of $K$ in $N_1, \ldots, N_4$. This provides an alternative to the classical principalization algorithm by Scholz and Taussky. The new algorithm, which is easily automatizable and executes very quickly, is implemented in PARI/GP and applied to all 4596, resp. 1146, quadratic fields with discriminant $-10^6 < D < 10^7$ and 3-class group of type $(3, 3)$, resp. $(9, 3)$, to obtain extensive statistics of their principalization types and the distribution of their second 3-class groups $G_3^2(K)$ on the coclass graphs $G(3, r)$, $1 \leq r \leq 6$, in the sense of Eick, Leedham-Green, and Newman.

References.


Introduction

The principal ideal theorem, which has been conjectured by Hilbert in 1898, states that each ideal of a number field $K$ becomes principal when it is extended to the Hilbert class field $F^1(K)$ of $K$, that is the maximal abelian unramified extension field of $K$. Inspired by the Artin-Furtwängler proof of the principal ideal theorem, Scholz and Taussky investigated the principalization in intermediate fields $K < N < F_3^1(K)$ between a base field $K$ with 3-class group of type $(3, 3)$ or $(9, 3)$ and its Hilbert 3-class field $F_3^1(K)$. They developed an algorithm for computing the principalization of $K$ in its four unramified cubic cyclic extension fields $N_1, \ldots, N_4$ for the case of a complex quadratic base field $K$. This algorithm is probabilistic, since it decides whether an ideal $a$ of $K$ becomes principal in $N_i$, for some $1 \leq i \leq 4$, by testing local cubic residue characters of a principal ideal cube $(a) = a^3$, associated with the ideal $a$, and of a fundamental unit $\xi_i$ of the non-Galois cubic subfield $L_i$ of the complex $S_3$-field $N_i$ with respect to a series of rational test primes $(p_t)_{t \in \mathbb{N}}$ and terminating when a critical test prime occurs. An upper bound for the minimal critical test prime $p_{\text{crit}}$ cannot be given effectively. It can only be estimated by means of Chebotarëv’s density theorem, thus causing uncertainty.

An entirely different approach to the principalization problem will be presented in this lecture. It is based on a purely group theoretical connection between the structure of the abelianizations $M_i/\gamma_2(M_i)$ of the four maximal normal subgroups $M_i$ of an arbitrary metabelian 3-group $G$ with $G/\gamma_2(G)$ of type $(3, 3)$ or $(9, 3)$ and the kernels $\ker(T_i)$ of the transfers $T_i : G/\gamma_2(G) \longrightarrow M_i/\gamma_2(M_i)$, $1 \leq i \leq 4$. By the Artin reciprocity law of class field theory, a corresponding number theoretical connection is established between the structure of the 3-class groups $\text{Cl}_3(N_i)$ of the four unramified cyclic cubic extension fields $N_i$ of an arbitrary algebraic number field $K$ with 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$ or $(9, 3)$ and the principalization kernels $\ker(j_{N_i/K})$ of the class extension homomorphisms $j_{N_i/K} : \text{Cl}_3(K) \longrightarrow \text{Cl}_3(N_i)$, $1 \leq i \leq 4$, applying the group theoretical statements to the second 3-class group $G_3^2(K) = \text{Gal}(F_3^2(K)/K)$ of $K$, that is the Galois group of the second Hilbert 3-class field $F_3^2(K) = F_3^1(F_3^1(K))$ of $K$. 
1. **Computational Performance of the Algorithm**

The history of determining principalization types of quadratic fields is shown in the following two tables 1 and 2.

**Table 1. History of investigating quadratic fields of type (3,3)**

<table>
<thead>
<tr>
<th>History authors</th>
<th>year</th>
<th>complex range</th>
<th>complex number</th>
<th>real range</th>
<th>real number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scholz, Taussky</td>
<td>1934</td>
<td>$-10^4 &lt; D$</td>
<td>2</td>
<td>$D &lt; 1 \cdot 10^5$</td>
<td>5</td>
</tr>
<tr>
<td>Heider, Schmithals</td>
<td>1982</td>
<td>$-20^4 &lt; D$</td>
<td>13</td>
<td>$D &lt; 2 \cdot 10^5$</td>
<td>16</td>
</tr>
<tr>
<td>Brink</td>
<td>1984</td>
<td>$-96^4 &lt; D$</td>
<td>66</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mayer</td>
<td>1989</td>
<td>$-30^4 &lt; D$</td>
<td>35</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mayer</td>
<td>1991</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mayer</td>
<td>2010</td>
<td>$-10^6 &lt; D$</td>
<td>2020</td>
<td>$D &lt; 10^7$</td>
<td>2576</td>
</tr>
</tbody>
</table>

**Table 2. History of investigating quadratic fields of type (9,3)**

<table>
<thead>
<tr>
<th>History authors</th>
<th>year</th>
<th>complex range</th>
<th>complex number</th>
<th>real range</th>
<th>real number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scholz, Taussky</td>
<td>1934</td>
<td>$-10^4 &lt; D$</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heider, Schmithals</td>
<td>1982</td>
<td>$-20^4 &lt; D$</td>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mayer</td>
<td>1989</td>
<td>$-30^4 &lt; D$</td>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mayer</td>
<td>2011</td>
<td>$-10^6 &lt; D$</td>
<td>875</td>
<td>$D &lt; 10^7$</td>
<td>271</td>
</tr>
</tbody>
</table>
2. Theoretical Foundations of the Algorithm

2.1. Little and big two-stage towers of 3-class fields.

$K$ an algebraic number field,
$\text{Cl}_3(K)$ its 3-class group of type (3, 3) or (9, 3),
$N_1, \ldots, N_4$ the four unramified cyclic cubic extensions of $K$,
$0 \leq \varepsilon \leq 4$ the counter $\#\{1 \leq i \leq 4 \mid \text{rank}_3(\text{Cl}_3(N_i)) \geq 3\}$,
$\tilde{N}_4 = \prod_{i=1}^{4} N_i$ the Frattini extension of $K$,
$F_1^3(K)$ the first Hilbert 3-class field of $K$,
$F_1^3(N_i)$ the first Hilbert 3-class field of $N_i$, $1 \leq i \leq 4$,
$F_2^3(K)$ the second Hilbert 3-class field of $K$.

**Definition 2.1.** For $1 \leq i \leq 4$,
the $\Gamma_i = \text{Gal}(F_1^3(N_i)|K)$ denote the Galois groups of the four little two-stage towers of $K$, $K < F_1^3(K) \leq F_1^3(N_i)$,
$G = \text{Gal}(F_2^3(K)|K)$ denotes the Galois group of the big two-stage tower of $K$, $K < F_1^3(K) \leq F_2^3(K)$.
$G = G_2^3(K)$ is called the second 3-class group of $K$. 
Further notation.

$\gamma_j(G)$, $j \geq 1$, the members of the lower central series of $G$,
$\chi_j(G)$, $j \geq 2$, the two-step centralizers of $\gamma_j(G)/\gamma_{j+2}(G)$,
$T_i : G/G' \rightarrow M_i/M'_i$ the transfer from $G$ to the
maximal subgroup $M_i$, $1 \leq i \leq 4$,
$\tau = \left(\text{str}(M_i/M'_i)\right)_{1 \leq i \leq 4}$ the transfer target type (TTT) of $G$,
where $M_i/M'_i \cong \text{Cl}_3(N_i)$,
$\varkappa = \left(\ker(T_i)\right)_{1 \leq i \leq 4}$ the transfer kernel type (TKT) of $G$,
briefly $\varkappa = \left(\varkappa(i)\right)_{1 \leq i \leq 4}$, where $\ker(T_i) = \tilde{M}_{\varkappa(i)}/G'$. 
2.2. Nearly homocyclic 3-class groups of 3-rank two.

**Definition 2.2.** For an integer $n \geq 2$, denote by $A(3, n)$ the nearly homocyclic abelian 3-group of order $3^n$, i.e., the abelian group of type $(3^{q+r}, 3^q)$, where $n = 2q + r$ with integers $q \geq 1$ and $0 \leq r < 2$. 
2.2.1. Second 3-class groups $G$ of coclass. $cc(G) = 1$

Figure 1. Root $C_3 \times C_3$ and branches $B(j)$, $2 \leq j \leq 7$, on the coclass graph $G(3,1)$

Order $3^m$

<table>
<thead>
<tr>
<th>TKT:</th>
<th>a.3</th>
<th>a.3</th>
<th>a.2</th>
<th>a.1</th>
<th>a.1</th>
<th>a.1</th>
<th>a.1</th>
</tr>
</thead>
</table>


Legend:
number in angles: GAP identifier,
big contour square: abelian group,
big full circle: metabelian group containing an abelian maximal subgroup,
small full circle: metabelian group without abelian maximal subgroups.
Theorem 2.1. (Transfer target type, TTT, for $G \in \mathcal{G}(3,1)$)

The structure of the 3-class groups $\text{Cl}_3(N_i)$, $1 \leq i \leq 4$, for $\text{cc}(G) = 1$, $|G| = 3^m$, $\text{cl}(G) = m - 1$, $m \geq 3$, is given by

$$\text{Cl}_3(N_1) \cong \begin{cases} A(3,m - 1), & \text{if } [\chi_2(G), \gamma_2(G)] = 1, \ m \geq 5, \\ A(3,m - 2), & \text{if } [\chi_2(G), \gamma_2(G)] = \gamma_{m-1}(G), \ m \geq 6, \end{cases}$$

$$\text{Cl}_3(N_i) \cong A(3,2) \ for \ 2 \leq i \leq 4, \ if \ m \geq 4.$$ 


Remark 2.1. Our Thm.2.1 covers all stem groups in $\Phi_9$, $\Phi_{35}$, $\Phi_{36}$, and in higher isoclinism families.

It remains to investigate two groups in $\Phi_2$, four groups in $\Phi_3$, and three groups in $\Phi_{10}$. 
2.2.2. *Second 3-class groups* $G$, $cc(G) \geq 2$, $G/G' \simeq (3, 3)$

**Figure 2.** Sporadic groups and roots of coclass trees on the coclass graph $G(3, 2)$

Order $3^n$

<table>
<thead>
<tr>
<th>9</th>
<th>$3^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>27</td>
<td>$3^3$</td>
</tr>
<tr>
<td>81</td>
<td>$3^4$</td>
</tr>
<tr>
<td>243</td>
<td>$3^5$</td>
</tr>
<tr>
<td>729</td>
<td>$3^6$</td>
</tr>
<tr>
<td>2187</td>
<td>$3^7$</td>
</tr>
<tr>
<td>6561</td>
<td>$3^8$</td>
</tr>
</tbody>
</table>

Edges of depth 2 forming the interface between $G(3, 1)$ and $G(3, 2)$

- $C_3 \times C_3$
- $G^3_0(0, 0)$

Legend:
- number in angles: GAP identifier,
- big contour square: abelian group,
- big full circle: metabelian group containing an abelian maximal subgroup,
- big contour circle: metabelian group with centre of type $(3, 3)$,
- small contour circle: metabelian group with centre of type $(3)$,
- small contour square: non-metabelian group.


TKT: D.10, D.5
(2241)(4224)

TKT: G.19, H.4
(2143)(4443)

TKT: b.10, c.18, c.21
(0043), (0313), (0231)
Theorem 2.2. (Partial TTT $(\tau_1(G), \tau_2(G))$ for $cc(G) \geq 2$)

The structure of the 3-class groups $\text{Cl}_3(N_i)$, $1 \leq i \leq 4$, for $cc(G) \geq 2$, $|G| = 3^n$, $\text{cl}(G) = m - 1$, $G/G' \simeq (3,3)$, and invariant $e = n - m + 2 \geq 3$, where $4 \leq m < n \leq 2m - 3$, is given by

\[
\begin{align*}
\text{Cl}_3(N_1) & \simeq \begin{cases} 
A(3, m - 1), & \text{if } [\chi_s(G), \gamma_e(G)] = 1, ~ m \geq 5, \\
A(3, m - 2), & \text{if } [\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G), ~ m \geq 6,
\end{cases} \\
\text{Cl}_3(N_2) & \simeq A(3, e) \text{ for } e \geq 4, \\
\text{Cl}_3(N_i) & \simeq A(3, 3) \text{ for } 3 \leq i \leq 4, \text{ if } \Gamma_i \not\simeq G_0^4(1,0) \simeq \text{Syl}_3A_9.
\end{align*}
\]


Remark 2.2. Our Thm.2.2 covers $\text{Cl}_3(N_i)$, $1 \leq i \leq 2$, for all stem groups in $\Phi_{23}$ and in higher isoclinism families.

It remains to investigate seven groups in $\Phi_6$, three groups in $\Phi_{40}$, three groups in $\Phi_{41}$, four groups in $\Phi_{42}$, and two groups in $\Phi_{43}$ completely,

$\text{Cl}_3(N_2)$, for $cc(G) = 2$,
and $\text{Cl}_3(N_i)$, $3 \leq i \leq 4$, for $cc(G) \geq 2$. 
2.3. Searching for 3-class groups of 3-rank three.

2.3.1. Second 3-class groups \( G \) of coclass \( cc(G) = 1 \).

**Theorem 2.3.** (TTT of stem groups \( G \) in \( \Phi_2, \Phi_3, \Phi_{10} \))

The structure of the 3-class groups \( \text{Cl}_3(N_i), 1 \leq i \leq 4 \), for \( cc(G) = 1, |G| = 3^m, 3 \leq \text{cl}(G) + 1 = m \leq 5 \), ist given by

\[
\text{Cl}_3(N_1) \cong \begin{cases} 
A(3, 2), & \text{if } m = 3, \\
A(3, 3), & \text{if } m = 4, \ G \not\cong G_0^4(1, 0) \cong \text{Syl}_3 A_9, \\
C_3 \times C_3 \times C_3, & \text{if } m = 4, \ G \cong G_0^4(1, 0) \cong \text{Syl}_3 A_9, \\
A(3, 3), & \text{if } k = 1, \ m = 5, 
\end{cases}
\]

\[
\text{Cl}_3(N_i) \cong \begin{cases} 
A(3, 2), & \text{if } G \cong G_0^3(0, 0), \\
C_9, & \text{if } G \cong G_0^3(0, 1), \text{ for } 2 \leq i \leq 4, \text{ if } m = 3,
\end{cases}
\]

where \( \text{Gal}(F_3^2(K)|N_1) = \langle y, \gamma_2(G) \rangle \) with \( y^3 = 1 \), for \( m = 3 \).

Corollary 2.3.1. \((TTT \tau(G)\) of stem groups \(G\) in \(\Phi_2, \Phi_3, \Phi_{10}\))

The following table gives the structure of the 3-class groups \(\text{Cl}_3(N_i), 1 \leq i \leq 4\), for the 3-groups \(G \in G(3,1)\) of small nilpotency class \(1 \leq \text{cl}(G) = m - 1 \leq 4\) in dependence on the principalisation or transfer kernel type, TKT, \(\kappa\).

\(\varepsilon\) counts 3-class groups of type \((3,3,3)\).

<table>
<thead>
<tr>
<th>(m)</th>
<th>(k)</th>
<th>TKT</th>
<th>(\kappa)</th>
<th>(\text{Cl}_3(F_3^1(K)))</th>
<th>(\text{Cl}_3(N_1))</th>
<th>(\text{Cl}_3(N_2))</th>
<th>(\text{Cl}_3(N_3))</th>
<th>(\text{Cl}_3(N_4))</th>
<th>(\varepsilon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>a.1</td>
<td>(0000)</td>
<td>1</td>
<td>(3)</td>
<td>(3)</td>
<td>(3)</td>
<td>(3)</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>a.1</td>
<td>(0000)</td>
<td>(3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>A.1</td>
<td>(1111)</td>
<td>(3)</td>
<td>(3,3)</td>
<td>(9)</td>
<td>(9)</td>
<td>(9)</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>a.1</td>
<td>(0000)</td>
<td>(3,3)</td>
<td>(9,3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>a.2</td>
<td>(1000)</td>
<td>(3,3)</td>
<td>(9,3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>a.3</td>
<td>(2000)</td>
<td>(3,3)</td>
<td>(9,3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>0</td>
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<tr>
<td>4</td>
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<td>a.3*</td>
<td>(2000)</td>
<td>(3,3)</td>
<td>(3,3,3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>a.1</td>
<td>(0000)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>(3,3)</td>
<td>0</td>
</tr>
</tbody>
</table>


Example 2.1. The smallest discriminant with TKT a.3*, where \(\varepsilon = 1\), is \(D = 142\,097\).
2.3.2. **Second 3-class groups** $G$, $\text{cc}(G) = 2$, $G/G' \simeq (3, 3)$.

**Theorem 2.4.** (TTT $\tau(G)$ of stem groups $G$ in $\Phi_6$)

If $|G| = 3^5$, $\text{cl}(G) = 3$, $G/G'' \simeq (3, 3)$, i.e., $G$ is one of the 7 top vertices on $G(3, 2)$ with bicyclic centre, then the structure of the 3-class groups of $F_3^1(K)$ and $N_1, \ldots, N_4$ is given by

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{TKT} & \hfill \kappa \hfill & \text{Cl}_3(F_3^1(K)) & \text{Cl}_3(N_1) & \text{Cl}_3(N_2) & \text{Cl}_3(N_3) & \text{Cl}_3(N_4) & \varepsilon \\
\hline
\text{D.10} & (2241) & (3, 3, 3) & (9, 3) & (9, 3) & (3, 3, 3) & (9, 3) & 1 \\
\text{D.5} & (4224) & (3, 3, 3) & (3, 3, 3) & (9, 3) & (3, 3, 3) & (9, 3) & 2 \\
\text{G.19} & (2143) & (3, 3, 3) & (9, 3) & (9, 3) & (9, 3) & (9, 3) & 0 \\
\text{H.4} & (4443) & (3, 3, 3) & (3, 3, 3) & (3, 3, 3) & (9, 3) & (3, 3, 3) & 3 \\
\hline
\text{b.10} & (0043) & (3, 3, 3) & (9, 3) & (9, 3) & (3, 3, 3) & (3, 3, 3) & 2 \\
\text{c.18} & (0313) & (3, 3, 3) & (9, 3) & (9, 3) & (3, 3, 3) & (9, 3) & 1 \\
\text{c.21} & (0231) & (3, 3, 3) & (9, 3) & (9, 3) & (9, 3) & (9, 3) & 0 \\
\hline
\end{array}
$$

in dependence on the transfer kernel type, TKT (principalisation type), $\kappa$ of $K$.

Here, $\varepsilon$ denotes the number of 3-class groups $\text{Cl}_3(N_i)$ of type $(3, 3, 3)$.


**Example 2.2.** The first discriminant with TKT D.10, where $\varepsilon = 1$, is $D = -4027$. 
**Theorem 2.5.** (TTT of stem groups $G$ in $\Phi_{40}, \Phi_{41}, \Phi_{42}, \Phi_{43}$) 

If $|G| = 3^6$, $\text{cl}(G) = 4$, $G/G' \simeq (3, 3)$, 

and $[\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G)$, i.e., 

$G$ is one of the 12 vertices of level 2 on coclass graph $G(3, 2)$ 

with cyclic centre, 

then the structure of the 3-class groups of $F_3^1(K)$ and $N_1, \ldots, N_4$ 

is given by

<table>
<thead>
<tr>
<th>TKT</th>
<th>$\kappa$</th>
<th>$\rho$</th>
<th>$\text{Cl}_3(F_3^1(K))$</th>
<th>$\text{Cl}_3(N_1)$</th>
<th>$\text{Cl}_3(N_2)$</th>
<th>$\text{Cl}_3(N_3)$</th>
<th>$\text{Cl}_3(N_4)$</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>G.19</td>
<td>(2143)</td>
<td>1</td>
<td>$(3, 3, 3, 3)$</td>
<td>$(9, 3)$</td>
<td>$(9, 3)$</td>
<td>$(9, 3)$</td>
<td>$(9, 3)$</td>
<td>0</td>
</tr>
<tr>
<td>H.4</td>
<td>(4443)</td>
<td>1</td>
<td>$(9, 3, 3)$</td>
<td>$(3, 3, 3)$</td>
<td>$(3, 3, 3)$</td>
<td>$(9, 3)$</td>
<td>$(3, 3, 3)$</td>
<td>3</td>
</tr>
<tr>
<td>b.10</td>
<td>(0043)</td>
<td>$-1$</td>
<td>$(3, 3, 3, 3)$</td>
<td>$(9, 3)$</td>
<td>$(9, 3)$</td>
<td>$(3, 3, 3)$</td>
<td>$(3, 3, 3)$</td>
<td>2</td>
</tr>
<tr>
<td>b.10</td>
<td>(0043)</td>
<td>1</td>
<td>$(9, 3, 3)$</td>
<td>$(9, 3)$</td>
<td>$(9, 3)$</td>
<td>$(3, 3, 3)$</td>
<td>$(3, 3, 3)$</td>
<td>2</td>
</tr>
</tbody>
</table>

in dependence on the TKT $\kappa$ of $K$ and on the relational exponent $\rho$ of $G$. 

Again, $\varepsilon$ denotes the number of 3-class groups $\text{Cl}_3(N_i)$ of type $(3, 3, 3)$.


**Example 2.3.** The first discriminant with TKT H.4, where $\varepsilon = 3$, is $D = -3896$. 

Theorem 2.6. (ε as a tree invariant)

All metabelian groups $G$ on the coclass tree $T(\langle 729, 40 \rangle)$, resp. $T(\langle 243, 6 \rangle)$, $T(\langle 243, 8 \rangle)$, of coclass graph $G(3, 2)$ are characterized by the value $\varepsilon = 2$, resp. $\varepsilon = 1$, $\varepsilon = 0$.

2.3.3. Second 3-class groups $G$, $cc(G) \geq 3$, $G/G' \simeq (3, 3)$.

**Theorem 2.7.** (TTT $\tau(G)$ for groups $G$ with $cc(G) \geq 3$) If $|G| = 3^n$, $n \geq 7$, $G/G'' \simeq (3, 3)$, $cc(G) \geq 3$, $m \leq n - 2$, $e \geq 4$, then $Cl_3(N_1)$, $Cl_3(N_2)$ are nearly homocyclic and $Cl_3(N_3)$, $Cl_3(N_4)$ are of type $(3, 3)$, independently from the transfer kernel type $\kappa$. In particular, $\varepsilon = 2$.

2.3.4. Second 3-class groups $G$, $cc(G) = 2$, $G/G' \simeq (9, 3)$

Figure 3. Sporadic groups and roots of coclass trees on the coclass graph $G(3, 2)$

Proof: J.A. Ascione, 1979, [As] On 3-groups of second maximal class.

Legend:
number in angles: GAP identifier,
big contour square: abelian group,
big full circle: metabelian group with centre of type $(3, 3)$,
big contour circle: metabelian group with centre of type $(9)$,
small contour circle: metabelian group with centre of type $(3)$.
Theorem 2.8. (TTT τ(\(G\)) of branch 1 groups \(G\) in \(\Phi_3\))

If \(|G| = 3^5\), \(\text{cl}(G) = 3\), \(G/G' \simeq (9,3)\), i.e., \(G\) is one of the 8 CF-groups of level 2 on coclass graph \(G(3,2)\),

then the structure of the 3-class groups of \(N_1, \ldots, N_4\) and the Frattini extension \(\tilde{N}_4\) is given by

<table>
<thead>
<tr>
<th>TKT</th>
<th>(\kappa)</th>
<th>(\text{Cl}_3(N_1))</th>
<th>(\text{Cl}_3(N_2))</th>
<th>(\text{Cl}_3(N_3))</th>
<th>(\text{Cl}_3(N_4))</th>
<th>(\varepsilon)</th>
<th>(\text{Cl}_3(\tilde{N}_4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>b.2</td>
<td>(0001)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(9,3,3)</td>
<td>1</td>
<td>(9,3)</td>
</tr>
<tr>
<td>b.3</td>
<td>(1000)</td>
<td>(27,3)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(3,3,3)</td>
<td>1</td>
<td>(9,3)</td>
</tr>
<tr>
<td>b.3</td>
<td>(1000)</td>
<td>(27,3)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(3,3,3)</td>
<td>1</td>
<td>(9,3)</td>
</tr>
<tr>
<td>b.16</td>
<td>(4000)</td>
<td>(9,3,3)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(3,3,3)</td>
<td>2</td>
<td>(3,3,3)</td>
</tr>
<tr>
<td>b.15</td>
<td>(0004)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(9,3,3)</td>
<td>1</td>
<td>(3,3,3)</td>
</tr>
<tr>
<td>b.15</td>
<td>(0004)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(3,3,3,3)</td>
<td>1</td>
<td>(3,3,3)</td>
</tr>
<tr>
<td>a.1</td>
<td>(0000)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(9,3,3)</td>
<td>1</td>
<td>(3,3,3)</td>
</tr>
<tr>
<td>a.1</td>
<td>(0000)</td>
<td>(9,3,3)</td>
<td>(9,3)</td>
<td>(9,3)</td>
<td>(3,3,3)</td>
<td>2</td>
<td>(3,3,3)</td>
</tr>
</tbody>
</table>

in dependence on the punctured TKT \(\kappa\) of \(K\).

Here, \(\varepsilon\) denotes the number of 3-class groups \(\text{Cl}_3(N_i)\) of rank greater than 2.


Example 2.4. The smallest discriminant with punctured TKT b.3, where \(\varepsilon = 1\), is \(D = 635\,909\).
2.3.5. Second 3-class groups $G$, $cc(G) = 3$, $G/G' \simeq (9, 3)$

**Figure 4.** Sporadic groups and roots of coclass trees on the coclass graph $G(3, 3)$

**Order** $3^n$

- $27$ ($3^3$)
- $81$ ($3^4$)
- $243$ ($3^5$)
- $729$ ($3^6$)
- $2187$ ($3^7$)
- $6561$ ($3^8$)
- $19683$ ($3^9$)
- $59049$ ($3^{10}$)

**TKT:**

- A.20
- B.15
- A.20
- C.31
- D.11
- d.10
- B.2
- D.10
- C.4
- E.12
- B.7
- e.14
- E.12
- D.6


Legend:
- Number in angles: GAP identifier,
- Big contour square: abelian group,
- Big full circle: metabelian group with centre of type $(3, 3)$,
- Medium contour square: metabelian group with centre of type $(3, 3, 3)$,
- Small contour square: metabelian group with centre of type $(9, 3)$,
- Small full circle: metabelian group of order at least 2187.
Theorem 2.9. \((TTT \tau(G) of branch 1 \ groups \ G \ in \ \Phi_6)\)

If \(|G| = 3^6\), \(cl(G) = 3\), \(G/G' \simeq (9,3)\), i.e.,
\(G\) is one of the 13 top vertices on coclass graph \(G(3,3)\),
then the structure of the 3-class groups of \(N_1, \ldots, N_4\) and
the Frattini extension \(\tilde{N}_4\) is given by

<table>
<thead>
<tr>
<th>TKT</th>
<th>(\kappa)</th>
<th>(\mathrm{Cl}_3(N_1))</th>
<th>(\mathrm{Cl}_3(N_2))</th>
<th>(\mathrm{Cl}_3(N_3))</th>
<th>(\mathrm{Cl}_3(N_4))</th>
<th>(\varepsilon)</th>
<th>(\mathrm{Cl}_3(\tilde{N}_4))</th>
</tr>
</thead>
<tbody>
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<td>((27,3))</td>
<td>((9,3,3))</td>
<td>2</td>
<td>((9,3,3))</td>
</tr>
<tr>
<td>D.11</td>
<td>(4322)</td>
<td>((9,3,3))</td>
<td>((27,3))</td>
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<td>((9,3,3))</td>
<td>2</td>
<td>((9,3,3))</td>
</tr>
<tr>
<td>B.7</td>
<td>(1114)</td>
<td>((27,3))</td>
<td>((27,3))</td>
<td>((27,3))</td>
<td>((9,3,3))</td>
<td>1</td>
<td>((9,3,3))</td>
</tr>
<tr>
<td>B.7</td>
<td>(1114)</td>
<td>((27,3))</td>
<td>((27,3))</td>
<td>((27,3))</td>
<td>((9,3,3))</td>
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<td>((9,3,3))</td>
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<tr>
<td>E.12</td>
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<tr>
<td>e.14</td>
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<td>((27,3))</td>
<td>((27,3))</td>
<td>((9,3,3))</td>
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<tr>
<td>e.14</td>
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<td>1</td>
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</tr>
<tr>
<td>A.20</td>
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<td>((9,3,3))</td>
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<td>((9,3,3))</td>
<td>((3,3,3))</td>
<td>4</td>
<td>((3,3,3))</td>
</tr>
<tr>
<td>b.15</td>
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<td>((9,3,3))</td>
<td>((9,3,3))</td>
<td>((9,3,3))</td>
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<td>4</td>
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</tr>
<tr>
<td>c.27</td>
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<td>((9,3,3))</td>
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<tr>
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<td>((9,3,3))</td>
<td>((9,3,3))</td>
<td>((9,3,3))</td>
<td>4</td>
<td>((3,3,3))</td>
</tr>
</tbody>
</table>

in dependence on the punctured TKT \(\kappa\) of \(K\).
Again, \(\varepsilon\) denotes the number of 3-class groups \(\mathrm{Cl}_3(N_i)\) of
rank greater than 2.


Example 2.5. The first discriminant with punctured TKT
D.11, where \(\varepsilon = 2\), is \(D = -3299\).
References.
[5] D. C. Mayer, *The structure of the 3-class groups of the four unramified cyclic cubic extensions of a number field with 3-class group of type $(3, 3)$* (Journées de Théorie des Nombres, Faculté des Sciences, Université Mohammed Premier, Oujda, Maroc, 2010).